

# Hodge correlators

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*To Alexander Beilinson for his 50th birthday*

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# 1 Introduction

## 1.1 Summary

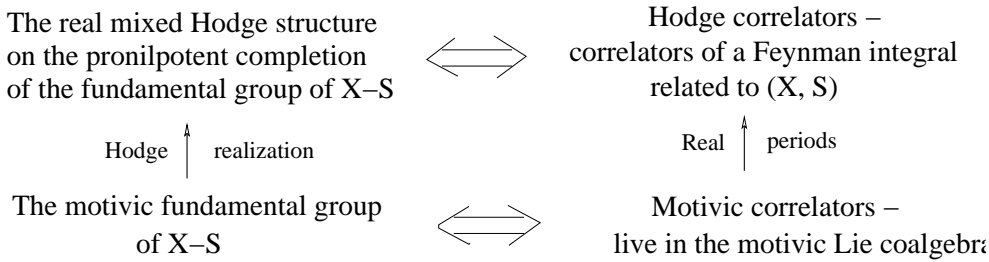
Let  $X$  be a smooth compact complex curve,  $v_0$  a non-zero tangent vector at a point  $s_0$  of  $X$ , and  $S^* = \{s_1, \dots, s_m\}$  a collection of distinct points of  $X$  different from  $s_0$ . We introduce *Hodge correlators* related to this datum. They are complex numbers, given by integrals of certain differential forms over products of copies of  $X$ . Let  $S = S^* \cup \{s_0\}$ . When the data  $(X, S, v_0)$  varies, Hodge correlators satisfy a system of non-linear quadratic differential equations. Using this, we show that Hodge correlators encode a variation of real mixed Hodge structures, called below mixed  $\mathbb{R}$ -Hodge structures. We prove that it coincides with the standard mixed  $\mathbb{R}$ -Hodge structure on the pronilpotent completion  $\pi_1^{\text{nil}}(X - S, v_0)$  of the fundamental group of  $X - S$ . The latter was defined, by different methods, by J. Morgan [M], R. Hain [H] and A.A. Beilinson

(cf. [DG]). The real periods of  $\pi_1^{\text{nil}}(X - S, v_0)$  have a well known description via Chen's iterated integrals. Hodge correlators give a completely different way to describe them.

Mixed Hodge structures are relatives of  $l$ -adic Galois representations. There are two equivalent definitions of mixed  $\mathbb{R}$ -Hodge structures: as vector spaces with the weight and Hodge filtrations satisfying some conditions [D], and as representations of the *Hodge Galois group* [D2].

The true analogs of  $l$ -adic representations are Hodge Galois group modules. However usually we describe mixed  $\mathbb{R}$ -Hodge structures arising from geometry by constructing the weight and Hodge filtrations, getting the Hodge Galois group modules only *a posteriori*. The Hodge correlators describe the mixed  $\mathbb{R}$ -Hodge structure on  $\pi_1^{\text{nil}}(X - S, v_0)$  directly as a module over the Hodge Galois group.

We introduce a Feynman integral related to  $X$ . It does not have a rigorous mathematical meaning. However the standard perturbative series expansion procedure provides a collection of its *correlators* assigned to the data  $(X, S, v_0)$ , which turned out to be convergent finite dimensional integrals. We show that they coincide with the Hodge correlators, thus explaining the name of the latter. Moreover:



We define *motivic correlators*. Their periods are the Hodge correlators. Motivic correlators lie in the *motivic Lie coalgebra*, and describe the motivic fundamental group of  $X - S$ . The coproduct in the motivic Lie coalgebra is a new feature, which is missing when we work just with numbers. We derive a simple explicit formula for the coproduct of motivic correlators. It allows us to perform *arithmetic analysis* of Hodge correlators. This is one of the essential advantages of the Hodge correlator description of the real periods of  $\pi_1^{\text{nil}}(X - S)$ , which has a lot of arithmetic applications. It was available before only for the rational curve case [G7].

The Lie algebra of the unipotent part of the Hodge Galois group is a free graded Lie algebra. Choosing a set of its generators we arrive at a collection of periods of a real MHS. We introduce new generators of the Hodge Galois group, which differ from Deligne's generators [D2]. The periods of variations mixed  $\mathbb{R}$ -Hodge structures corresponding to these generators satisfy non-linear quadratic Maurer-Cartan type differential equations. For the subcategory of Hodge-Tate structures they were defined by A. Levin [L]. The Hodge correlators are the periods for this set of generators.

We introduce a DG Lie coalgebra  $\mathcal{L}_{\mathcal{H}, X}^*$ . The category of  $\mathcal{L}_{\mathcal{H}, X}^*$ -comodules is supposed to be a DG-enhancement of the category of *smooth  $\mathbb{R}$ -Hodge sheaves*, i.e. the subcategory of Saito's mixed  $\mathbb{R}$ -Hodge sheaves whose cohomology are variations of mixed  $\mathbb{R}$ -Hodge structures. We show that the category of comodules over the Lie coalgebra  $H^0 \mathcal{L}_{\mathcal{H}, X}^*$  is equivalent to the category of variations of mixed  $\mathbb{R}$ -Hodge structures.

The simplest Hodge correlators for the rational and elliptic curves deliver single-valued versions of the classical polylogarithms and their elliptic counterparts, the classical Eisenstein-Kronecker series [We]. The latter were interpreted by A.A. Beilinson and A. Levin [BL] as periods of variations of mixed  $\mathbb{R}$ -Hodge structures. More generally, when  $X = \mathbb{CP}^1$  the Hodge correlators are real periods of variations of mixed Hodge structures related to multiple polylogarithms. For an elliptic curve  $E$  they deliver the multiple Eisenstein-Kronecker series defined in [G1].

When  $X$  is a modular curve and  $S$  is the set of its cusps, Hodge correlators generalize the Rankin-Selberg integrals. Indeed, one of the simplest of them is the Rankin-Selberg convolution of a pair  $f_1, f_2$  of weight two cuspidal Hecke eigenforms with an Eisenstein series. It computes, up to certain constants, the special value  $L(f_1 \times f_2, 2)$ . A similar Hodge correlator gives the Rankin-Selberg convolution of a weight two cuspidal Hecke eigenform  $f$  and two Eisenstein series, and computes, up to certain constants,  $L(f, 2)$ . (A generalization to higher weight modular forms will appear elsewhere).

The simplest motivic correlators on modular curves deliver Beilinson's elements in motivic cohomology, e.g. the Beilinson-Kato Euler system in  $K_2$ . We use motivic correlators to define *motivic multiple  $L$ -values* related to: Dirichlet characters of  $\mathbb{Q}$  (for  $X = \mathbb{C}^* - \mu_N$ , where  $\mu_N$  is the group of  $N$ -th roots of unity); Hecke Grössencharacters imaginary quadratic fields (for CM elliptic curves minus torsion points); Jacobi Grössencharacters of cyclotomic fields (for affine Fermat curves); the weight two modular forms.

The Hodge correlators considered in this paper admit a generalization when  $X - S$  is replaced by an arbitrary regular complex projective variety. We prove that they describe the mixed  $\mathbb{R}$ -Hodge structure on the rational motivic homotopy type of  $X$ , defined by a different method in [M]. We describe them in a separate paper since the case of curves is more transparent, and so far has more applications<sup>1</sup>.

Key constructions of this paper were outlined in Sections 8-9 of [G1], which may serve as an introduction.

## 1.2 Arithmetic motivation I: special values of $L$ -functions

Beilinson's conjectures imply that special values of  $L$ -functions of motives over  $\mathbb{Q}$  at the integral points to the left of the critical line are periods of mixed motives over  $\mathbb{Q}$ . This means that they can be written as integrals

$$\int_{\Delta_B} \omega_A$$

where  $A$  and  $B$  are normal crossing divisors over  $\mathbb{Q}$  in a smooth  $n$ -dimensional projective variety  $X$  over  $\mathbb{Q}$ ,  $\omega_A \in \Omega_{\log}^n(X - A)$  is an  $n$ -form with logarithmic singularities at  $A$ , and  $\Delta_B$  is an  $n$ -chain with the boundary on  $B(\mathbb{C})$ . Moreover, the special values are periods of their motivic avatars – the motivic  $\zeta$ -elements – which play a crucial role in arithmetic applications.

The classical example is given by the Leibniz formula for the special values of the Riemann  $\zeta$ -function:

$$\zeta(n) = \int_{0 \leq t_1 \leq \dots \leq t_n \leq 1} \frac{dt_1}{1 - t_1} \wedge \frac{dt_2}{t_2} \wedge \dots \wedge \frac{dt_n}{t_n}.$$

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<sup>1</sup>The case of a smooth compact Kahler manifold was treated in [GII]

The corresponding motivic  $\zeta$ -elements were extensively studied ([BK], [BD], [HW]). Beilinson's conjectures are known for the special values of the Dedekind  $\zeta$ -function of a number field thanks to the work of Borel [Bo1], [Bo2].

Here is another crucial example. Let  $f(z)$  be a weight two cuspidal Hecke eigenform. So  $f(z)dz$  is a holomorphic 1-form on a compactified modular curve  $\overline{M}$ . Recall that by the Manin-Drinfeld theorem the image in the Jacobian of any degree zero cuspidal divisor  $a$  on  $\overline{M}$  is torsion. So there is a modular unit  $g_a \in \mathcal{O}^*(M)$  such that  $\text{div}(g_a)$  is a multiple of the divisor  $a$ . According to Bloch and Beilinson [B] we get an element

$$\{g_a, g_b\} \in K_2(\overline{M}) \otimes \mathbb{Q}. \quad (1)$$

Applying the regulator map to this element, and evaluating it on the 1-form  $f(z)dz$ , we get an integral

$$\int_{\overline{M}(\mathbb{C})} \log |g_a| \overline{\partial} \log |g_b| f(z) dz. \quad (2)$$

It is a Rankin-Selberg convolution integral. Indeed,  $\log |g_a|$  is the value of a non-holomorphic Eisenstein series at  $s = 1$ , and  $\partial \log |g_b|$  is a holomorphic weight two Eisenstein series. Therefore according to the Rankin - Selberg method, the integral is proportional to the product of special values of  $L(f, s)$  at  $s = 1$  and  $s = 2$ . Moreover (see [SS] for a detailed account) one can find cuspidal divisors  $a$  and  $b$  such that the proportionality coefficient is non-zero, i.e.

$$\int_{\overline{M}(\mathbb{C})} \log |g_a| \overline{\partial} \log |g_b| f(z) dz \sim L(f, 2). \quad (3)$$

Using the functional equation for  $L(f, s)$ , one can easily deduce that  $L'(f, 0)$  is a period. Finally, suitable modifications of Beilinson's  $\zeta$ -elements (1) give rise to Kato's Euler system [Ka].

There are similar results, due to Beilinson [B2] in the weight two case, and Beilinson (unpublished) and, independently, Scholl (cf. [DS]) for the special values of L-functions of cuspidal Hecke eigenforms of arbitrary weight  $w \geq 2$  at any integral point to the left of the critical strip.

This picture, especially the motivic  $\zeta$ -elements, seem to come out of the blue. Furthermore, already for  $L''(\text{Sym}^2 f, 0)$ , related by the functional equation to  $L(\text{Sym}^2 f, 3)$  we do not know in general how to prove that it is a period.

One may ask whether there is a general framework, which delivers naturally both Rankin-Selberg integrals (2) and Beilinson's  $\zeta$ -elements (1), and tells where to look for generalizations related to non-critical special values  $L(\text{Sym}^m f, n)$ .

We suggest that one should look at the motivic fundamental group of the universal modular curve. We show that the simplest Hodge correlator for the modular curve coincides with the Rankin-Selberg integral (2), and the corresponding motivic correlator is Beilinson's motivic  $\zeta$ -element (1) – see Section 1.10 for an elaborate discussion.

We will show elsewhere that the Rankin-Selberg integrals related to the values at non-critical special points of L-functions of arbitrary cuspidal Hecke eigenforms and the corresponding motivic  $\zeta$ -elements appear naturally as the Hodge and motivic correlators related to the standard local systems on the modular curve: the present paper deals with the trivial local system.

### 1.3 Arithmetic motivation II: arithmetic analysis of periods

The goal of *arithmetic analysis* is to investigate periods without actually computing them, using instead arithmetic theory of mixed motives.<sup>2</sup> Here is the simplest non-trivial example, provided by the values of the dilogarithm at rational numbers. Recall that the classical dilogarithm

$$\mathrm{Li}_2(z) := - \int_0^z \log(1-t) \frac{dt}{t}$$

is a multivalued function on  $\mathbb{CP}^1 - \{0, 1, \infty\}$ . Its value at  $z$  depends on the homotopy class of the path from 0 to  $z$  on  $\mathbb{CP}^1 - \{0, 1, \infty\}$  used to define the integral. Computing the monodromy of the dilogarithm, we see that  $\mathrm{Li}_2(z)$  is well defined modulo the subgroup of  $\mathbb{C}$  generated by  $(2\pi i)^2$  and  $2\pi i \log(z)$ . So the values  $\mathrm{Li}_2(z)$  at  $z \in \mathbb{Q}$  are well defined modulo the subgroup  $2\pi i \log \mathbb{Q}^*$  spanned by  $\mathbb{Z}(2) := (2\pi i)^2 \mathbb{Z}$  and  $2\pi i \log q$ ,  $q \in \mathbb{Q}^*$  – the latter is well defined modulo  $\mathbb{Z}(2)$ .

Consider a map of abelian groups

$$\Delta : \mathbb{Z}[\mathbb{Q}^* - \{1\}] \longrightarrow \mathbb{Q}^* \otimes \mathbb{Q}^*, \quad \{z\} \longmapsto -(1-z) \otimes z. \quad (4)$$

**Conjecture 1.1** *Let  $z_i \in \mathbb{Q}$ ,  $a_j, b_j \in \mathbb{Q}^*$  and  $n_i \in \mathbb{Z}$ . Then one has*

$$\sum_i n_i \mathrm{Li}_2(z_i) + \sum_j \log a_j \cdot \log b_j = 0 \pmod{(2\pi i \log \mathbb{Q}^*)} \quad (5)$$

*if and only if*

$$\Delta\left(\sum_i n_i \{z_i\}\right) + \sum_j (a_j \otimes b_j + b_j \otimes a_j) = 0 \quad \text{in } (\mathbb{Q}^* \otimes \mathbb{Q}^*) \otimes_{\mathbb{Z}} \mathbb{Q}. \quad (6)$$

This gives a complete conjectural description of the  $\mathbb{Q}$ -linear relations between the values of the dilogarithm and the products of two logarithms at rational numbers.

**Theorem 1.2** *The condition (6) implies (5).*

This tells the identities between integrals without actually computing the integrals.

**Example.** One has

$$\mathrm{Li}_2\left(\frac{1}{3}\right) + \mathrm{Li}_2\left(-\frac{1}{2}\right) + \frac{1}{2}(\log \frac{3}{2})^2 = 0 \pmod{(2\pi i \log \mathbb{Q}^*)}. \quad (7)$$

Indeed, modulo 2-torsion in  $\mathbb{Q}^* \otimes \mathbb{Q}^*$  (notice that  $3/2 \otimes (-1)$  is a 2-torsion) one has:

$$\left(1 - \frac{1}{3}\right) \otimes \frac{1}{3} + \left(1 + \frac{1}{2}\right) \otimes \left(-\frac{1}{2}\right) = \frac{2}{3} \otimes \frac{1}{3} + \frac{3}{2} \otimes \frac{1}{2} = \frac{3}{2} \otimes \frac{3}{2}. \quad (8)$$

---

<sup>2</sup>The latter so far mostly is a conjectural theory, so we often arrive to conjectures rather than theorems.

**Motivic avatars of the logarithm and the dilogarithm.** The values of the dilogarithm and the product of two logarithms at rational arguments are weight two mixed Tate periods over  $\mathbb{Q}$ . In fact they describe all such periods. Here is the precise meaning of this.

The category of mixed Tate motives over a number field  $F$  is canonically equivalent to the category of graded comodules over a Lie coalgebra  $\mathcal{L}_\bullet(F)$  over  $\mathbb{Q}$ , graded by positive integers, the weights (cf. [DG]). The graded Lie coalgebra  $\mathcal{L}_\bullet(F)$  is free, with the space of generators in degree  $n$  isomorphic to  $K_{2n-1}(F) \otimes \mathbb{Q}$ . Equivalently, the kernel of the coproduct map

$$\delta : \mathcal{L}_\bullet(F) \longrightarrow \Lambda^2 \mathcal{L}_\bullet(F)$$

is isomorphic to the graded space  $\oplus_{n>0} K_{2n-1}(F) \otimes \mathbb{Q}$ , where  $K_{2n-1}(F) \otimes \mathbb{Q}$  is in degree  $n$ . One has  $K_1(F) = F^*$ , and the group  $K_{2n-1}(F)$  is of finite rank, which is  $r_1 + r_2$  for odd and  $r_2$  for even  $n$  [Bo2].<sup>3</sup> It follows that  $\mathcal{L}_1(F) = F^* \otimes_{\mathbb{Z}} \mathbb{Q}$ . Let us describe the  $\mathbb{Q}$ -vector space  $\mathcal{L}_2(F)$ .

For any field  $K$ , consider a version of the map (4):

$$\delta : \mathbb{Z}[K^* - \{1\}] \longrightarrow K^* \wedge K^*, \quad \{z\} \longmapsto -(1-z) \wedge z.$$

Let  $r(*, *, *, *)$  be the cross-ratio of four points on  $\mathbb{P}^1$ , normalized by  $r(\infty, 0, 1, x) = x$ . Let  $R_2(K)$  be the subgroup of  $\mathbb{Z}[K^* - \{1\}]$  generated by the elements

$$\sum_{i=1}^5 (-1)^i \{r(x_1, \dots, \hat{x}_i, \dots, x_5)\}, \quad x_i \in P^1(F), \quad x_i \neq x_j. \quad (9)$$

Then  $\delta(R_2(K)) = 0$ . Let  $B_2(K) := \mathbb{Z}[K^* - \{1\}]/R_2(K)$  be the Bloch group of  $K$ . The following proposition is deduced in Section 1 of [G10] from a theorem of Suslin [S].

**Proposition 1.3** *Let  $F$  be a number field. Then one has  $\mathcal{L}_2(F) = B_2(F) \otimes_{\mathbb{Z}} \mathbb{Q}$ .*<sup>4</sup>

The dual of the universal enveloping algebra of Lie coalgebra  $\mathcal{L}_\bullet(F)$  is a commutative graded Hopf algebra. Its weight two component is isomorphic to is

$$\mathcal{L}_2(F) \oplus \text{Sym}^2 \mathcal{L}_1(F) = (B_2(F) \oplus \text{Sym}^2 F^*) \otimes_{\mathbb{Z}} \mathbb{Q}. \quad (10)$$

The coproduct is provided by the map

$$\Delta_2 : B_2(F) \oplus \text{Sym}^2 F^* \longrightarrow (F^* \otimes F^*) \otimes_{\mathbb{Z}} \mathbb{Q} \quad (11)$$

$$\{z\}_2 \longmapsto \frac{1}{2} \left( z \otimes (1-z) - (1-z) \otimes z \right), \quad a \cdot b \longmapsto a \otimes b + b \otimes a.$$

Let us return to our example, when  $F = \mathbb{Q}$ . The period homomorphism

$$\mathcal{L}_1(\mathbb{Q}) \longrightarrow \mathbb{C}/\mathbb{Z}(1), \quad q \longmapsto \log q$$

is known to be injective. So  $q \in \mathbb{Q}^*$  is the motivic avatar of  $\log q$ .

<sup>3</sup>Here  $r_1$  and  $r_2$  are the numbers of real and complex places of  $F$ , so that  $[F : \mathbb{Q}] = 2r_2 + r_1$ .

<sup>4</sup>The same should be true for any field  $F$ . However the Lie coalgebra  $\mathcal{L}_\bullet(F)$  in that case is a conjectural object.

**Proposition 1.4** *There is a weight two period map*

$$\begin{aligned} \mathcal{L}_2(\mathbb{Q}) \oplus \mathrm{Sym}^2 \mathcal{L}_1(\mathbb{Q}) &\longrightarrow \mathbb{C}/2\pi i \log \mathbb{Q}^*, \\ \{q\}_2 &\longmapsto \mathrm{Li}_2(q) + \frac{1}{2} \log(1-q) \log(q), \quad a \otimes b + b \otimes a \longmapsto \log a \log b. \end{aligned} \tag{12}$$

The point is that the period map kills the subgroup  $R_2(\mathbb{C})$ , which is equivalent to Abel's five term relation for the dilogarithm.

The period map (12) is conjectured to be injective: this is a special case of a Grothendieck type conjecture on periods. Finally,  $\mathrm{Ker} \Delta_2 = K_3(\mathbb{Q}) \otimes \mathbb{Q}$ , which is zero by Borel's theorem. The injectivity of  $\Delta_2$  plus Propositions 1.3 and 1.4 imply Theorem 1.2, and, modulo the injectivity of the period map (12), Conjecture 1.1.

Although relation (7) follows from the five term relation for the dilogarithm, there is no procedure to write the element  $\{\frac{1}{3}\} + \{-\frac{1}{2}\}$  as a sum of the five term relations (9). The only effective way to know that they exist is to calculate the coproduct, as we did in (8).

Let us now discuss the case when  $F$  is an arbitrary number field.

*Real periods.* There is a single valued version of the dilogarithm, the Bloch-Wigner function

$$\mathcal{L}_2(z) := -\mathrm{Im} \left( \int_0^z \log(1-t) \frac{dt}{t} + \int_0^z \frac{dt}{1-t} \cdot \log |z| \right).$$

where both integrals are defined by using the same integration path from 0 to  $z$ . It satisfies the five term relation, and thus provides a group homomorphism

$$B_2(\mathbb{C}) \longrightarrow \mathbb{R}, \quad \{z\}_2 \rightarrow \mathcal{L}_2(z).$$

Using the (conjectural) isomorphism  $\mathcal{L}_2(\mathbb{C}) = B_2(\mathbb{C})$ , it can be interpreted as the *real period map*  $\mathcal{L}_2(\mathbb{C}) \rightarrow \mathbb{R}$ . Notice that  $\mathcal{L}_2(z) = 0$  for  $z \in \mathbb{R}$ . So the real period map loses a lot of information about the motivic Lie algebra. It relates, however, the kernel of the coproduct map with the special values of  $L$ -functions. Here is how it works in our running example.

Let  $F$  be a number field. Then by Borel's theorem [Bo2] the real period map provides an injective regulator map

$$K_3(F) \otimes \mathbb{Q} \xrightarrow{\cong} \mathrm{Ker} \delta \subset B_2(F) \otimes \mathbb{Q} \longrightarrow \mathbb{R}^{r_2}, \quad \{z\}_2 \longmapsto \left( \mathcal{L}_2(\sigma_1(z)), \dots, \mathcal{L}_2(\sigma_{r_2}(z)) \right).$$

Its image is a rational lattice – that is, a lattice tensor  $\mathbb{Q}$  – in  $\mathbb{R}^{r_2}$ , whose covolume, well defined up to  $\mathbb{Q}^*$ , is a  $\mathbb{Q}^*$ -multiple of  $\zeta_F(-1)$ .

So to perform the arithmetic analysis of the values of the dilogarithm / products of two logarithms at rational arguments we upgrade them to their motivic avatars lying in (10), and determine their coproduct there. The kernel of the coproduct is captured by the regulator map.

The weight  $m$  periods of mixed Tate motives over a number field  $F$  are studied similarly using the iterated coproduct in  $\otimes^m F^*$ , see Section 4 of [G7].

In this paper we develop a similar picture for the periods of the pronilpotent completions of fundamental groups of curves. Namely, we introduce motivic correlators, which are the motivic avatars of the periods of fundamental groups of curves. They span a Lie subcoalgebra in the motivic Lie coalgebra. The key point is that the coproduct of a motivic correlator is given explicitly in terms of the motivic correlators. We prove that their real periods are given by the Hodge correlators.



## 1.4 Pronilpotent completions of fundamental groups of curves

The fundamental group  $\pi_1 = \pi_1(X - S, v_0)$  is a free group with generators provided by loops around the rest of the punctures  $s_i \neq s_0$  and loops generating  $H_1(X)$ . Let  $\mathcal{I} := \text{Ker}(\mathbb{Q}[\pi] \rightarrow \mathbb{Q})$  be the augmentation ideal of the group algebra of  $\pi_1$ . Then there is a complete cocommutative Hopf algebra over  $\mathbb{Q}$

$$A^{\text{Betti}}(X - S, v_0) := \varprojlim \mathbb{Q}[\pi_1]/\mathcal{I}^n. \quad (13)$$

Its coproduct is induced by the map  $g \mapsto g \otimes g$ ,  $g \in \pi_1$ . It is called the *fundamental Hopf algebra of  $X - S$*  with the tangential base point  $v_0$ . The subset of its primitive elements is a free pronilpotent Lie algebra over  $\mathbb{Q}$ , the Maltsev completion of  $\pi_1$ . It is denoted  $\pi_1^{\text{nil}}(X - S, v_0)$  and called the *fundamental Lie algebra of  $X - S$* . The Hopf algebra (13) is its universal enveloping algebra.

Denote by  $T(V)$  the tensor algebra of a vector space  $V$ . The associated graded for the  $\mathcal{I}$ -adic filtration is isomorphic to the tensor algebra of  $H_1(X - S, \mathbb{Q})$ :

$$\text{gr}^{\mathcal{I}} A^{\text{Betti}}(X - S, v_0) = \bigoplus_{n=0}^{\infty} \mathcal{I}^n / \mathcal{I}^{n+1} = T(H_1(X - S, \mathbb{Q})).$$

There is a weight filtration on  $H_1(X - S, \mathbb{Q})$ , given by the extension

$$0 \longrightarrow \mathbb{Q}(1)^{|S|-1} \longrightarrow H_1(X - S, \mathbb{Q}) \longrightarrow H_1(X, \mathbb{Q}) \longrightarrow 0, \quad (14)$$

where the right arrow is provided by the embedding  $X - S \hookrightarrow X$ . The Hopf algebra (13) is equipped with a weight filtration  $W$  compatible with the Hopf algebra structure. The corresponding associated graded of the Hopf algebra (13) is isomorphic to the tensor algebra of  $\text{gr}^W H_1(X - S, \mathbb{Q})$ :

$$\text{gr}^W A^{\text{Betti}}(X - S, v_0) = T(\text{gr}^W H_1(X - S, \mathbb{Q})). \quad (15)$$

## 1.5 Hodge correlators

Recall  $S^* = S - \{s_0\}$ . Set

$$V_{X, S^*} := H_1(X, \mathbb{C}) \oplus \mathbb{C}[S^*], \quad A_{X, S^*} := T(V_{X, S^*}).$$

One sees from (14), (15) that there are canonical isomorphisms of vector spaces

$$V_{X, S^*} = \text{gr}^W H_1(X - S, \mathbb{C}), \quad A_{X, S^*} = \text{gr}^W A^{\text{Betti}}(X - S, v_0) \otimes \mathbb{C}.$$

We also need the dual objects:

$$V_{X, S^*}^{\vee} := H^1(X, \mathbb{C}) \oplus \mathbb{C}[S^*], \quad A_{X, S^*}^{\vee} := T(V_{X, S^*}^{\vee}).$$

For an associative algebra  $A$  over  $k$ , let  $A^+ := \text{Ker}(A \rightarrow k)$  be the augmentation ideal. Let  $[A^+, A^+]$  be the subspace (not the ideal) of  $A^+$  generated by commutators in  $A^+$ . The quotient space  $\mathcal{C}(A) := A^+ / [A^+, A^+]$  is called the *cyclic envelope* of  $A$ . If  $A$  is freely generated by a set  $\mathcal{S}$  then the vector space  $\mathcal{C}(A)$  has a basis parametrised by cyclic words in  $\mathcal{S}$ .

We use the shorthand  $\mathcal{CT}(V)$  for  $\mathcal{C}(T(V))$ . We define the subspace of *shuffle relations* in  $\mathcal{CT}(V)$  as the subspace generated by the elements

$$\sum_{\sigma \in \Sigma_{p,q}} (v_0 \otimes v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(p+q)}), \quad p, q \geq 1,$$

where the sum is over all  $(p, q)$ -shuffles. Set

$$\mathcal{C}_{X,S^*}^\vee := \mathcal{C}(A_{X,S^*}^\vee), \quad \mathcal{CLie}_{X,S^*}^\vee := \frac{\mathcal{C}_{X,S^*}^\vee}{\text{Shuffle relations}}; \quad (16)$$

$$\mathcal{C}_{X,S^*} := \mathcal{C}(A_{X,S^*}) \quad \mathcal{CLie}_{X,S^*} := \text{the dual of } \mathcal{CLie}_{X,S^*}^\vee. \quad (17)$$

We show that  $\mathcal{C}_{X,S^*}^\vee$  is a Lie coalgebra,  $\mathcal{CLie}_{X,S^*}^\vee$  is its quotient Lie coalgebra. Equivalently,  $\mathcal{C}_{X,S^*}$  is a Lie algebra,  $\mathcal{CLie}_{X,S^*}$  is its Lie subalgebra. .

Given a mass one volume form  $\mu$  on  $X$ , our Feynman integral construction in Section 2 provides a linear map, called the *Hodge correlator map*

$$\text{Cor}_{\mathcal{H},\mu} : \mathcal{CLie}_{X,S^*}^\vee \longrightarrow \mathbb{C}. \quad (18)$$

**Example.** For a cyclic word

$$W = \mathcal{C}\left(\{a_0\} \otimes \{a_1\} \otimes \dots \otimes \{a_n\}\right), \quad a_i \in S^*, \quad (19)$$

the Hodge correlator is given by a sum over all plane trivalent trees  $T$  whose external edges are decorated by elements  $a_0, \dots, a_n$ , see Fig 1. To define the integral corresponding to such a tree  $T$  we proceed as follows. The volume form  $\mu$  provides a Green function  $G(x, y)$  on  $X^2$ . Each edge  $E$  of  $T$  contributes a Green function on  $X^{\{\text{vertices of } E\}}$ , which we lift to a function on  $X^{\{\text{vertices of } T\}}$ . Further, given any  $m+1$  smooth functions on a complex manifold  $M$ , there is a canonical linear map  $\omega_m : \Lambda^{m+1} \mathcal{A}_M^0 \rightarrow \mathcal{A}_M^m$ , where  $\mathcal{A}_M^k$  is the space of smooth  $k$ -forms on  $M$  (Section 2.2). Applying it to the Green functions assigned to the edges of  $T$ , we get a differential form of the top degree on  $X^{\{\text{internal vertices of } T\}}$ . Integrating it, we get the integral assigned to  $T$ . Taking the sum over all trees  $T$ , we get the number  $\text{Cor}_{\mathcal{H},\mu}(W)$ . The shuffle relations result from taking the sum over all trees with a given decoration.

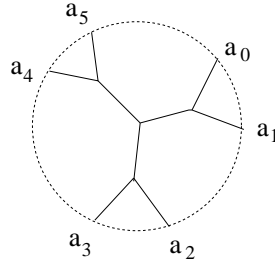


Figure 1: A plane trivalent tree decorated by  $\mathcal{C}(\{a_0\} \otimes \dots \otimes \{a_5\})$ .

Dualising the Hodge correlator map (18), we get a *Green operator*

$$\mathbb{G}_\mu \in \mathcal{CLie}_{X,S^*}. \quad (20)$$

Let  $L_{X,S^*}$  be the free Lie algebra generated by the space  $V_{X,S^*}$ . There is a canonical isomorphism

$$L_{X,S^*} = \text{gr}^W \pi_1^{\text{nil}}(X - S, v_0) \otimes \mathbb{C}. \quad (21)$$

The Green operator can be viewed as a derivation of the Lie algebra  $L_{X,S^*}$  as follows.

Let  $(p_i, q_i)$  be a symplectic basis of  $H_1(X)$ . Denote by  $X_s$  the generator of  $L_{X,S^*}$  assigned to  $s \in S^*$ . There is a canonical generator of  $\text{gr}^W \pi_1^{\text{nil}}(X - S, v_0)$  corresponding to a loop around  $s_0$ . Its projection to  $L_{X,S^*}$  is written as

$$X_{s_0} := - \sum_{s \in S^*} X_s + \sum [p_i, q_i]. \quad (22)$$

Say that a derivation of the Lie algebra  $L_{X,S^*}$  is *special* if it kills the generator (22) and preserves the conjugacy classes of the generators  $X_s$ , where  $s \in S^*$ .

This definition is motivated by the  $l$ -adic picture (Section 9.2): If  $X$  is a curve over a field  $F$  and  $S \subset X(F)$ , the Galois group  $\text{Gal}(\overline{F}/F(\mu_{l^\infty}))$  acts by special automorphisms of  $L_{X,S^*} \otimes \mathbb{Q}_l$ . Indeed, by the comparison theorem the latter is identified with the associate graded for the weight filtration on the Lie algebra of the pro- $l$ -completion  $\pi_1^{(l)}(X - S, v_0)$  of the fundamental group.

Denote the Lie algebra of special derivations by  $\text{Der}^S L_{X,S^*}$ . We show that the Lie algebra  $\mathcal{CLie}_{X,S^*}$  acts by special derivations on the Lie algebra  $L_{X,S^*}$ . This generalizes constructions of Drinfeld [Dr1] (when  $X$  is of genus zero) and Kontsevich [K] (when  $S$  is empty). Namely, let  $F$  be a cyclic polynomial in non-commuting variables  $Y_i$ , i.e.  $F \in \mathcal{CT}(Y)$  where  $Y$  is a vector space with a basis  $\{Y_i\}$ . Then there are “partial derivatives” maps

$$\partial/\partial Y_i : \mathcal{CT}(Y) \longrightarrow \mathcal{T}(Y), \quad F \longmapsto \partial F / \partial Y_i,$$

defined by deleting one of  $Y_i$ ’s from  $F$ , getting as a result a non-commutative polynomial, and taking the sum over all possibilities. For instance if  $F = \mathcal{C}(Y_1 Y_2 Y_1 Y_3)$  then  $\partial F / \partial Y_1 = Y_2 Y_1 Y_3 + Y_3 Y_1 Y_2$ .

Let  $F$  be a cyclic polynomial in non-commuting variables  $X_s, p_i, q_i$ , where  $s \in S^*$ . Then the derivation  $\kappa_F$  assigned to  $F$  acts on the generators by

$$p_i \longmapsto -\frac{\partial F}{\partial q_i}, \quad q_i \longmapsto \frac{\partial F}{\partial p_i}, \quad X_s \longmapsto [X_s, \frac{\partial F}{\partial X_s}], \quad s \in S^*.$$

*A priori* it is a derivation of the associative algebra  $A_{X,S^*}$ . However if  $F$  annihilates the shuffle relations, it is a derivation of the Lie algebra  $L_{X,S^*}$ . We show that we get an isomorphism

$$\mathcal{CLie}_{X,S^*} \xrightarrow{\sim} \text{Der}^S L_{X,S^*}, \quad F \longmapsto \kappa_F. \quad (23)$$

Combining (20) and (23), we conclude that the Green operator can be viewed as an element

$$\mathbb{G}_\mu \in \text{Der}^S L_{X,S^*}. \quad (24)$$

We show that it encodes a mixed  $\mathbb{R}$ -Hodge structure using a construction outlined in Section 1.7.

## 1.6 A Feynman integral for Hodge correlators

Let  $\varphi$  be a smooth function on  $X$  with values in  $N \times N$  complex matrices. We say that  $\{\varphi\}$  is our space of fields. Given  $N$ , consider the following correlator corresponding to  $W \in \mathcal{C}_{X,S}^\vee$ , formally defined via a Feynman integral

$$\text{Cor}_{X,N,h}(W) := \int \mathcal{F}_W(\varphi) e^{iS(\varphi)} \mathcal{D}\varphi, \quad (25)$$

where

$$S(\varphi) := \frac{1}{2\pi i} \int_X \text{Tr} \left( \frac{1}{2} \partial \varphi \wedge \bar{\partial} \varphi + \frac{1}{6} \hbar \cdot \varphi [\partial \varphi, \bar{\partial} \varphi] \right).$$

and  $\mathcal{F}_W(\varphi)$  is a function on the space of fields, see Section 12. For example, for the cyclic word (19) we set

$$\mathcal{F}_W(\varphi) := \text{Tr} \left( \varphi(a_0) \dots \varphi(a_n) \right). \quad (26)$$

Formula (25) does not have a precise mathematical meaning. We understand it by postulating the perturbative series expansion with respect to a small parameter  $\hbar$ , using the standard Feynman rules, and then taking the leading term in the asymptotic expansion as  $N \rightarrow \infty$ ,  $\hbar = N^{-1/2}$ . This way we get a sum over a finite number of Feynman diagrams given by plane trivalent trees decorated by the factors of  $W$ . One needs to specify a volume form  $\mu$  on  $X$  to determine a measure  $\mathcal{D}\varphi$  on the space of fields. Mathematically, we need  $\mu$  to specify the Green function which is used to write the perturbative series expansion.

We prove in Theorem 12.4 that these Feynman integral correlators coincide with the suitably normalised Hodge correlators.

**Problem.** The Hodge correlators provide only the leading term of the  $N \rightarrow \infty$  asymptotics of the Feynman integral correlators. The next terms in the asymptotics are given by the correlators corresponding to higher genus ribbon graphs. They are typically divergent. Can they be renormalised? What is their role in the Hodge theory?

## 1.7 Variations of mixed $\mathbb{R}$ -Hodge structures by twistor connections

A *real Hodge structure* is a real vector space  $V$  whose complexification is equipped with a bigrading  $V_{\mathbb{C}} = \oplus V^{p,q}$  such that  $\overline{V}^{p,q} = V^{q,p}$ . A real Hodge structure is *pure* if  $p+q$  is a given number, called the weight. So a real Hodge structure is a direct sum of pure ones of different weights. The category of real mixed Hodge structures is equivalent to the category of representations of the algebraic group  $\mathbb{C}_{\mathbb{R}}^*$  (i.e. the group  $\mathbb{C}^*$  understood as a real group).

According to P. Deligne [D], a *mixed  $\mathbb{R}$ -Hodge structure* is a real vector space  $V$  equipped with a weight filtration  $W_{\bullet}$ , and a Hodge filtration  $F^{\bullet}$  of its complexification  $V_{\mathbb{C}}$ , satisfying the following condition. The filtration  $F^{\bullet}$  and its conjugate  $\overline{F}^{\bullet}$  induce on  $\text{gr}_n^W V$  a pure weight  $n$  real Hodge structure:

$$\text{gr}_n^W V_{\mathbb{C}} = \oplus_{p+q=n} F_{(n)}^p \cap \overline{F}_{(n)}^q.$$

Here  $F_{(n)}^{\bullet}$  is the filtration on  $\text{gr}_n^W V_{\mathbb{C}}$  induced by  $F^{\bullet}$ , and similarly  $\overline{F}_{(n)}^{\bullet}$ .

The category of mixed  $\mathbb{R}$ -Hodge structures is a Tannakian category with a canonical fiber functor  $\omega : H \mapsto \text{gr}^W H$  to the category of  $\mathbb{R}$ -Hodge structures. Thus, thanks to the Tannakian formalism, it is canonically equivalent to the category of representations of a pro-algebraic group, the *Hodge Galois group*  $G_{\text{Hod}}$ . The group  $G_{\text{Hod}}$  is a semidirect product

$$0 \longrightarrow U_{\text{Hod}} \longrightarrow G_{\text{Hod}} \longrightarrow \mathbb{C}_{\mathbb{R}}^* \longrightarrow 0.$$

of a real prounipotent algebraic group  $U_{\text{Hod}}$  and  $\mathbb{C}_{\mathbb{R}}^*$ . There is an embedding  $s : \mathbb{C}_{\mathbb{R}}^* \hookrightarrow G_{\text{Hod}}$  corresponding to the functor  $H \rightarrow \text{gr}^W H$ . Let  $L_{\text{Hod}}$  be the Lie algebra of the group  $U_{\text{Hod}}$ . The action of  $s(\mathbb{C}_{\mathbb{R}}^*)$  provides the Lie algebra  $L_{\text{Hod}}$  with a structure of a Lie algebra in the category of  $\mathbb{R}$ -Hodge structures. The category of mixed  $\mathbb{R}$ -Hodge structures is equivalent to the category

of representations of the Lie algebra  $L_{\text{Hod}}$  in the category of  $\mathbb{R}$ -Hodge structures. The Lie algebra  $L_{\text{Hod}}$  is a free Lie algebra in the category of  $\mathbb{R}$ -Hodge structures. Thus the Lie algebra  $L_{\text{Hod}}$  is generated by the  $\mathbb{R}$ -Hodge structure

$$\bigoplus_{[H]} \text{Ext}_{\mathbb{R}\text{-MHS}}^1(\mathbb{R}(0), H)^\vee \otimes H.$$

where the sum is over the set of isomorphism classes of simple  $\mathbb{R}$ -Hodge structures  $H$ . The ones with non-zero  $\text{Ext}^1$  are parametrised by pairs of negative integers  $(-p, -q)$  up to a permutation (the Hodge degrees of  $H$ ), and one has  $\dim_{\mathbb{R}} \text{Ext}_{\mathbb{R}\text{-MHS}}^1(\mathbb{R}(0), H) = 1$  for those  $H$ .

It follows that the complex Lie algebra  $L_{\text{Hod}} \otimes_{\mathbb{R}} \mathbb{C}$  is generated by certain elements  $n_{p,q}$ ,  $p, q \geq 1$ , of bidegree  $(-p, -q)$ , with the only relation  $\bar{n}_{p,q} = -n_{q,p}$ . Such a generators were defined by Deligne [D2]. We call them Deligne's generators.

This just means that a mixed  $\mathbb{R}$ -Hodge structure can be described by a pair  $(V, g)$ , where  $V$  is a real vector space whose complexification is equipped with a bigrading  $V_{\mathbb{C}} = \bigoplus V^{p,q}$  such that  $\bar{V}^{p,q} = V^{q,p}$ , and  $g = \sum_{p,q \geq 1} g_{p,q}$  is an imaginary operator on  $V_{\mathbb{C}}$ , so that  $g_{p,q}$  is of bidegree  $(-p, -q)$  and  $\bar{g}_{p,q} = -g_{q,p}$ . The operators  $g_{p,q}$  are the images of Deligne's generators  $n_{p,q}$  in the representation  $V$ . One can describe variations of real Hodge structures using the operators  $g_{p,q}$  fiberwise, but the Griffiths transversality condition is encoded by complicated nonlinear differential equations on  $g_{p,q}$ .

A.A. Beilinson emphasized that Deligne's generators may not be the most natural ones, and asked whether one should exist a canonical choice of the generators. For the subcategory of mixed  $\mathbb{R}$ -Hodge-Tate structures (i.e. when the Hodge numbers are zero unless  $p = q$ ) a different set of generators  $n_p$  was suggested by A. Levin [L].

Let us decompose the Hodge correlator  $\mathbb{G} = \sum \mathbb{G}_{p,q}$  according to the Hodge bigrading. Then it is tempting to believe that the elements  $\mathbb{G}_{p,q}$  are images of certain generators of the Lie algebra  $L_{\text{Hod}}$ , thus providing the mixed  $\mathbb{R}$ -Hodge structure on  $\pi_1^{\text{nil}}(X - S, v_0)$ . Furthermore, we proved (Section 6) that, when the data  $(X, S, v_0)$  varies, the components  $\mathbb{G}_{p,q}$  satisfy a Maurer-Cartan type nonlinear quadratic differential equations.

Motivated by this, we show that a variation of mixed  $\mathbb{R}$ -Hodge structures can be described by a direct sum of variations of pure Hodge structures plus a *Green datum*  $\{G_{p,q}, \nu\}$  consisting of operator-valued functions  $G_{p,q}$ ,  $p, q \geq 1$  and a 1-form  $\nu$ , satisfying a system of quadratic nonlinear differential equations. The Green operators  $G_{p,q}$  are nonlinear modifications of Deligne's operators  $g_{p,q}$ , providing a canonical set of generators of the Hodge Lie algebra  $L_{\text{Hod}}$ . For the Hodge-Tate structures we recover the Green operators of [L], although even in this case our treatment is somewhat different, based on the twistor transform defined in Section 5.

Let us describe our construction. Let  $\mathcal{V}$  be a direct sum of variations of pure real Hodge structures on  $X$ . It is given by a real local system  $\mathcal{V}$ , whose complexification  $\mathcal{V}_{\mathbb{C}}$  has a canonical decomposition

$$\mathcal{V}_{\mathbb{C}} = \bigoplus_{p,q} \mathcal{V}^{p,q}. \quad (27)$$

Tensoring  $\mathcal{V}$  with the sheaf of smooth complex functions on  $X$ , we get a  $C^\infty$ -bundle  $\mathcal{V}_\infty$  with a flat connection  $\mathbf{d}$  satisfying the Griffiths transversality condition. Pick a real closed 1-form

$$\nu \in \Omega^1 \otimes \text{End}^{-1,0} \mathcal{V}_\infty \oplus \bar{\Omega}^1 \otimes \text{End}^{0,-1} \mathcal{V}_\infty, \quad \mathbf{d}\nu = 0, \quad \bar{\nu} = \nu,$$

and a collection of imaginary operators

$$G_{p,q} \in \text{End}^{-p,-q} \mathcal{V}_\infty, \quad p, q \geq 1, \quad \bar{G}_{p,q} = -G_{p,q}.$$

Let  $\mathbf{d} = \partial' + \partial''$  be the decomposition into the holomorphic and antiholomorphic components. Set  $\mathbf{d}^c = \partial' - \partial''$ . Consider the following 1-form with values in the local system  $\text{End}\mathcal{V}$ :

$$\psi := \mathbf{d}^c G + \nu.$$

Let  $\varphi_{p,q}^{a,b}$  be an  $\text{End}^{-p,-q}\mathcal{V}$ -valued  $(a,b)$ -form. We define its *Dolbeaux bigrading*  $(s,t)$  by setting  $s := a - p$  and  $t := b - q$ . Denote by  $\varphi_{s,t}$  the component of  $\varphi$  of the Dolbeaux bidegree  $(-s, -t)$ .

**Twistor connections.** Consider a “twistor line”  $\mathbb{R}$  with a coordinate  $u$ , the product  $X \times \mathbb{R}$ , and the canonical projection  $p : X \times \mathbb{R} \rightarrow X$ . Starting from a datum  $\{G_{p,q}, \nu\}$ , we introduce a *twistor connection*  $\nabla_{\mathcal{G}}$  on  $p^*\mathcal{V}_{\infty}$ :

$$\nabla_{\mathcal{G}} := \mathbf{d} + \sum_{s,t \geq 0} (1-u)^s (1+u)^t \left( (s+t+1)G_{s+1,t+1} du + \psi_{s,t} \right). \quad (28)$$

The structure of the twistor connection can be clarified by writing it in a homogeneous form as follows. Consider a “twistor plane”  $\mathbb{C}^2$  with coordinates  $(z,w)$ , the product  $X \times \mathbb{C}^2$ , and the projection  $\pi : X \times \mathbb{C}^2 \rightarrow X$ . The *homogeneous twistor connection* is a connection on  $\pi^*\mathcal{V}_{\infty}$ :

$$\widehat{\nabla}_{\mathcal{G}} := \mathbf{d} + \sum_{s,t \geq 0} z^s w^t \left( (s+t+1)G_{s+1,t+1} (zdw - wdz)/2 + \psi_{s,t} \right). \quad (29)$$

The group  $\mathbb{C}^* \times \mathbb{C}^*$  acts on the twistor plane:  $(z,w) \mapsto (\lambda_1 z, \lambda_2 w)$ , as well as on the Dolbeaux complex of the local system  $\text{End}\mathcal{V}$  on  $X$ , providing the Dolbeaux bigrading:  $\varphi_{s,t} \mapsto \lambda_1^{-s} \lambda_2^{-t} \varphi_{s,t}$ . The connection  $\widehat{\nabla}_{\mathcal{G}}$  is invariant under the action of the group  $\mathbb{C}^* \times \mathbb{C}^*$ . Restricting to “complex twistor line”  $z + w = 2$  parametrised by  $z = 1 - u, w = 1 + u$ , we recover (28).

The twistor line  $\mathbb{R}$  is the set of fixed points of the involution  $\sigma : (z,w) \mapsto (\bar{w}, \bar{z})$  of the twistor plane  $\mathbb{C}^2$ , restricted to the line  $z + w = 2$ , so in fact  $u \in i\mathbb{R}$ . The connection  $\widehat{\nabla}_{\mathcal{G}}$  is invariant under the composition of the involution  $\sigma$  with the complex conjugation  $c$ . Moreover,

$$(c \circ \sigma)^* \widehat{\nabla}_{\mathcal{G}} = \widehat{\nabla}_{\mathcal{G}} \text{ if and only if } \overline{G}_{p,q} = -G_{p,q}, \quad \overline{\nu} = \nu.$$

We show that  $\nabla_{\mathcal{G}}$  is flat if and only if the datum  $\{G_{p,q}, \nu\}$  satisfies a Maurer-Cartan type system of quadratic nonlinear differential equations. We call it then a *Green datum*. Green data, or, equivalently, flat twistor connections, form an abelian tensor category in an obvious way.

**Theorem 1.5** *The abelian tensor category of flat twistor connections is canonically equivalent to the category of variations of mixed  $\mathbb{R}$ -Hodge structures on  $X$ .*

A flat twistor connection gives rise to a variation of mixed  $\mathbb{R}$ -Hodge structures on  $X$  as follows. Restricting  $\nabla_{\mathcal{G}}$  to  $X \times \{0\}$  we get a flat connection  $\nabla_{\mathcal{G}}^0$  on  $\mathcal{V}_{\infty}$ . We equip the local system  $(\mathcal{V}_{\infty}, \nabla_{\mathcal{G}}^0)$  with a structure of a variation of mixed  $\mathbb{R}$ -Hodge structures. The weight filtration is the standard weight filtration on a bigraded object:

$$W_n \mathcal{V} := \oplus_{p,q \leq n} \mathcal{V}^{p,q}.$$

Let  $P$  be the operator of parallel transport for the connection  $\nabla_{\mathcal{G}}$  along the  $\mathbb{R}$ -factor from  $X \times \{1\}$  to  $X \times \{0\}$ . Take the standard Hodge filtration on the bigraded object  $F_{\text{st}}^p \mathcal{V}_{\mathbb{C}} := \oplus_{i \geq p} \mathcal{V}^{i,*}$  on the

restriction of  $p^*\mathcal{V}$  to  $X \times \{1\}$ . We define the Hodge filtration  $F^\bullet$  on  $\mathcal{V}_\mathbb{C}$  by applying the operator of parallel transport  $P$  to the standard Hodge filtration on  $p^*\mathcal{V}_\mathbb{C}$  at  $u = 1$ :

$$F^p\mathcal{V}_\mathbb{C} := P(F_{\text{st}}^p\mathcal{V}_\mathbb{C}).$$

We prove that  $(\mathcal{V}_\infty, \nabla_{\mathcal{G}}^0, W_\bullet, F^\bullet)$  is a variation of real mixed Hodge structures on  $X$ , and that any variation is obtained this way.

Just recently M. Kapranov [Ka] gave a neat interpretation of the category of mixed  $\mathbb{R}$ -Hodge structures as the category of  $\mathbb{C}_{\mathbb{C}/\mathbb{R}}^*$ -equivariant (not necessarily flat) connections on  $\mathbb{C}$ . His description is equivalent to the one by twistor connections when  $X$  is a point. Precisely, homogeneous twistor connections (29), restricted to  $\mathbb{C} = (\mathbb{C}^2)^\sigma$ , are natural representatives of the gauge equivalence classes of  $\mathbb{C}_{\mathbb{C}/\mathbb{R}}^*$ -equivariant connections on  $\mathbb{C}$ .

## 1.8 The twistor transform, Hodge DG algebra, and variations of real MHS

Theorem 1.5 describes the category of variations of mixed  $\mathbb{R}$ -Hodge structures via twistor connections. The next step would be to describe complexes of variations of mixed  $\mathbb{R}$ -Hodge structures. Precisely, we want an alternative description of the subcategory  $\text{Sh}_{\text{Hod}}^{\text{sm}}(X)$  of Saito's (derived) category  $\text{Sh}_{\text{Hod}}(X)$  of mixed  $\mathbb{R}$ -Hodge sheaves on  $X$  consisting of complexes whose cohomology are smooth, i.e. are variations of mixed  $\mathbb{R}$ -Hodge structures. We are going to define a DG Lie coalgebra, and conjecture that the DG-category of comodules over it is a DG-enhancement of the triangulated category of smooth mixed  $\mathbb{R}$ -Hodge sheaves on a compact complex manifold  $X$ . To include the case of a smooth open complex variety  $X$ , one needs to impose conditions on the behavior of the forms forming the DG Lie coalgebra at infinity.

*The semi-simple tensor category  $\text{Hod}_X$ .* Let  $X$  be a regular complex projective variety. Denote by  $\text{Hod}_X$  the category of variations of real Hodge structures on  $X(\mathbb{C})$ . Its objects are direct sums of variations of real pure Hodge structures of possibly different weights. The category  $\text{Hod}_X$  is a semi-simple abelian tensor category. Denote by  $w(\mathcal{L})$  the weight of a pure variation  $\mathcal{L}$ . The commutativity morphism is given by  $\mathcal{L}_1 \otimes \mathcal{L}_2 \longrightarrow (-1)^{w(\mathcal{L}_1)w(\mathcal{L}_2)} \mathcal{L}_2 \otimes \mathcal{L}_1$ .

*Hodge complexes.* Given a variation  $\mathcal{L}$  of pure Hodge structures on  $X$ , we introduce a complex  $\mathcal{C}_{\mathcal{H}}^\bullet(\mathcal{L})$  and show that it calculates  $\text{RHom}_{\text{Sh}_{\text{Hod}}(X)}(\mathbb{R}(0), \mathcal{L})$  in the category of mixed  $\mathbb{R}$ -Hodge sheaves on  $X$ . We call it the *Hodge complex* of  $\mathcal{L}$ . When  $\mathcal{L} = \mathbb{R}(n)$  it is the complex calculating the weight  $n$  Beilinson-Deligne cohomology of  $X$ .

The smooth de Rham complexes of variations of real Hodge structures on  $X$  can be organized into a commutative DG algebra in the category  $\text{Hod}_X$ , called the de Rham DGA:

$$\mathcal{A}_X := \oplus_{\mathcal{L}} \mathcal{A}^\bullet(\mathcal{L}) \bigotimes \mathcal{L}^\vee,$$

where  $\mathcal{L}$  runs through the isomorphism classes of simple objects in the category  $\text{Hod}_X$ . The product is given by the wedge product of differential forms and tensor product of variations.

We are going to define a commutative DG algebra  $\mathcal{D}_X$  in the category  $\text{Hod}_X$ . As an object, it is a direct sum of complexes

$$\mathcal{D}_X := \oplus_{\mathcal{L}} \mathcal{C}_{\mathcal{H}}^\bullet(\mathcal{L}) \bigotimes \mathcal{L}^\vee, \tag{30}$$

where  $\mathcal{L}$  runs through the isomorphism classes of simple objects in the category  $\text{Hod}_X$ . To define a DGA structure on  $\mathcal{D}_X$ , we recall the projection  $p : X \times \mathbb{R} \rightarrow X$ , and introduce our main hero:

*The twistor transform.* It is a linear map

$$\gamma : \text{The Hodge complex of } \mathcal{V} \text{ on } X \longrightarrow \text{The de Rham complex of } p^*\mathcal{V} \text{ on } X \times \mathbb{R},$$

Its definition is very similar to the formula (28) / (29) for the twistor connection – see Definition 3.3. The twistor transform is evidently injective, and gives rise to an injective linear map

$$\gamma : \mathcal{D}_X \hookrightarrow \mathcal{A}_X. \quad (31)$$

Our key result in Section 3 is the following theorem.

**Theorem 1.6** *Let  $X$  be a complex manifold. Then the image of the twistor transform (31) is closed under the differential and product, and thus is a DG subalgebra of the de Rham DGA.*

Theorem 1.6 provides  $\mathcal{D}_X$  with a structure of a commutative DGA in the category  $\text{Hod}_X$ . We call it the *Hodge DGA* of the complex variety  $X$ . The differential  $\delta$  on  $\mathcal{D}_X$  is obtained by conjugation by a degree-like operator  $\mu$ , see (82), of the natural differential on (30).

It is surprising that one can realize the Hodge DGA inside of the de Rham DGA on  $X \times \mathbb{R}$ .

Given a semi-simple abelian tensor category  $\mathcal{P}$ , denote by  $DGCom_{\mathcal{P}}$  and  $DGCoLie_{\mathcal{P}}$  the categories of DG commutative and Lie coalgebras in the category  $\mathcal{P}$ . Recall the two standard functors

$$\mathcal{B} : DGCom_{\mathcal{P}} \longrightarrow DGCoLie_{\mathcal{P}}, \quad \mathcal{C} : DGCoLie_{\mathcal{P}} \longrightarrow DGCom_{\mathcal{P}}.$$

The functor  $\mathcal{B}$  is the bar construction followed by projection to the indecomposables. The functor  $\mathcal{C}$  is given by the Chevalley standard complex:

$$(\mathcal{G}^\bullet, d) \longmapsto \mathcal{C}(\mathcal{G}) := (\text{Sym}^*(\mathcal{G}^\bullet[1]), \delta), \quad \delta := d_{Ch} + d.$$

Applying the functor  $\mathcal{B}$  to the commutative DGA  $\mathcal{D}_X$  in the category  $\text{Hod}_X$  we get a DG Lie coalgebra  $\mathcal{L}_{\mathcal{H};X}^* := \mathcal{B}(\mathcal{D}_X)$  in the same category. Let  $\mathcal{L}_{\mathcal{H};X} := H^0(\mathcal{L}_{\mathcal{H};X}^*)$  be the Lie coalgebra given by its zero cohomology. We show that Theorem 1.5 is equivalent to

**Theorem 1.7** *The category of comodules over the Lie coalgebra  $\mathcal{L}_{\mathcal{H};X}$  in the category  $\text{Hod}_X$  is canonically equivalent to the category of variations of mixed  $\mathbb{R}$ -Hodge structures on  $X$ . The equivalence is given by the functor  $\text{gr}^W$  of the associate graded for the weight filtration.*

When  $X$  is a point it provides a canonical set of generators for the Lie algebra of the Hodge Galois group. The DGA  $\mathcal{D}_X$  contains as a sub DGA the Hodge-Tate algebra of A. Levin [L], and our description of variations of Hodge-Tate structures essentially coincides with the one in *loc. cit.*

**Conjecture 1.8** *The category  $\text{Sh}_{\text{Hod}}^{\text{sm}}(X)$  of smooth complexes of real Hodge sheaves on  $X$  is equivalent to the DG-category of DG-modules over the DG Lie coalgebra  $\mathcal{L}_{\mathcal{H};X}^*$ .*



### 1.9 Mixed $\mathbb{R}$ -Hodge structure on $\pi_1^{\text{nil}}$ via Hodge correlators

Recall that  $L_{X,S^*}$ , see (21), is a free Lie algebra generated by  $\text{gr}^W H^1(X - S)$ . When the data  $(X, S, v_0)$  varies, the latter forms a variation  $\mathcal{L}$  of real Hodge structures over the enhanced moduli space  $\mathcal{M}'_{g,n}$ ,  $n = |S|$ , (we add a tangent vector  $v_0$  to the standard data). We are going to equip it with a Green datum, thus getting a variation mixed  $\mathbb{R}$ -Hodge structures.

A point  $s_0 \in X$  determines a  $\delta$ -current given by the evaluation of a test function at  $s_0$ . Viewed as a generalized volume form  $\mu$ , it provides a Green function  $G(x, y)$  up to a constant. We use the tangent vector  $v_0$  at  $s_0$  to specify the constant. Namely, let  $t$  be a local parameter at  $s_0$  with  $\langle dt, v_0 \rangle = 1$ . We normalize the Green function so that  $G(s_0, s) - \log |t|$  vanishes at  $s = s_0$ .

Let  $\mathbb{G}_{v_0}$  be the corresponding Green operator (24). It is an endomorphism of the smooth bundle  $\mathcal{L}_\infty$ . Thus the construction described in Section 1.7 provides a collection of mixed  $\mathbb{R}$ -Hodge structures in the fibers of  $\mathcal{L}$ .

Furthermore, the Hodge correlators in families deliver in addition to the Green operator  $\mathbb{G}_{v_0}$  an operator valued 1-form  $\nu$ . We prove in Section 6 that the pair  $\mathbf{G}_{v_0} := (\mathbb{G}_{v_0}, \nu)$  satisfies the Green data differential equations. Therefore we get a variation of mixed  $\mathbb{R}$ -Hodge structures.

**Theorem 1.9** *When the data  $(X, S, v_0)$  varies, the variation of mixed  $\mathbb{R}$ -Hodge structures given by the Green datum  $\mathbf{G}_{v_0}$  is isomorphic to the variation formed by the standard mixed  $\mathbb{R}$ -Hodge structures on  $\pi_1^{\text{nil}}(X - S, v_0)$ .*

The standard MHS on  $\pi_1^{\text{nil}}(X - S, v_0)$  is described by using Chen's theory of iterated integrals [Ch]. Our approach is different: it is given by integrals of non-holomorphic differential forms over products of copies of  $X$ . Theorem 1.9 implies that the two sets of periods obtained in these two descriptions coincide. For example the periods of the real MHS on  $\pi_1^{\text{nil}}(\mathbb{P}^1 - \{0, 1, \infty\}, v_0)$ , where  $v_0 = \partial/\partial t$  at  $t = 0$ , are given by the multiple  $\zeta$ -values [DG]. So Theorem 1.9 implies that the Hodge correlators in this case are  $\mathbb{Q}$ -linear combinations of the multiple  $\zeta$ -values, which is far from being obvious from their definition.

Here is another benefit of our approach: for a modular curve  $X$ , it makes obvious that the Rankin-Selberg convolutions are periods of the motivic fundamental group of  $X - \{\text{the cusps}\}$ .

### 1.10 Motivic correlators on curves

Let  $X$  be a regular projective curve over a field  $F$ ,  $S$  a non-empty subset of  $X(F)$ , and  $v_0$  a tangent vector at a point  $s_0 \in S$  defined over  $F$ . The *motivic fundamental group* of  $X - S$  with the tangential base point  $v_0$  is supposed to be a pro-unipotent Lie algebra  $L(X - S, v_0)$  in the hypothetical abelian category of mixed motives over  $F$ .

At the moment we have it either in realizations,  $l$ -adic or Hodge [D1], or when  $X = \mathbb{P}^1$  and  $F$  is a number field [DG]. We work in one of these settings, or assume the existence of the abelian category of mixed motives. A precise description of possible settings see in Section 9.2.

Let us use the conjectural motivic setting. The Tannakian category of mixed motives is canonically equivalent to the category of modules over a Lie algebra  $L_{\text{Mot}/F}$  in the category of pure motives over  $\text{Spec}(F)$ . We also use notation  $L_{\text{Mot}}$ . The equivalence is provided by the fiber functor  $M \mapsto \text{gr}^W M$ . Denote by  $\mathcal{L}_{\text{Mot}/F}$  the dual Lie coalgebra. So there is an isomorphism

$$H^i(\mathcal{L}_{\text{Mot}/F}) = \oplus_{[M]} \text{Ext}_{L_{\text{Mot}/F}}^i(\mathbb{Q}(0), M) \otimes M^\vee. \quad (32)$$

where the sum is over the isomorphism classes of pure motives. In particular, one has

$$\mathrm{Ker}\left(\mathcal{L}_{\mathrm{Mot}/\mathbb{F}} \xrightarrow{\delta} \Lambda^2 \mathcal{L}_{\mathrm{Mot}/\mathbb{F}}\right) = \oplus_{[M]} \mathrm{Ext}_{\mathrm{Mot}/\mathbb{F}}^1(\mathbb{Q}(0), M) \otimes M^\vee. \quad (33)$$

Since  $L(X - S, v_0)$  is a pro-object in the category of mixed motives, the Lie algebra  $L_{\mathrm{Mot}/\mathbb{F}}$  acts by special derivations on its associate graded for the weight filtration:

$$L_{\mathrm{Mot}/\mathbb{F}} \longrightarrow \mathrm{Der}^S\left(\mathrm{gr}^W L(X - S, v_0)\right). \quad (34)$$

Starting from the pure motive  $\mathrm{gr}^W H_1(X - S)$  rather than from its Betti realization, and following (17), we define the motivic version  $\mathcal{C}\mathcal{L}ie_{\mathcal{M};X,S^*}$  of  $\mathcal{C}\mathcal{L}ie_{X,S^*}$ . It is a Lie algebra in the semisimple abelian category of pure motives. Let  $\mathcal{C}\mathcal{L}ie_{\mathcal{M};X,S^*}^\vee$  be the dual Lie coalgebra. Thanks to (23), there is a Lie algebra isomorphism (which gets a Tate twist in the motivic setting)

$$\mathrm{Der}^S\left(\mathrm{gr}^W L(X - S, v_0)\right) = \mathcal{C}\mathcal{L}ie_{\mathcal{M};X,S^*}(-1).$$

So dualising (34), we arrive at a map of Lie coalgebras, called the *motivic correlator map*:

$$\mathrm{Cor}_{\mathrm{Mot}} : \mathcal{C}\mathcal{L}ie_{\mathcal{M};X,S^*}^\vee(1) \longrightarrow \mathcal{L}_{\mathrm{Mot}/\mathbb{F}}. \quad (35)$$

The left hand side is decomposed into a direct sum of cyclic tensor products of simple pure motives. Their images under the motivic correlator map are called *motivic correlators*.

If  $F$  is a number field, then, conjecturally,  $H^i(\mathcal{L}_{\mathrm{Mot}/\mathbb{F}}) = 0$  for  $i > 1$ . So the motivic Lie algebra  $L_{\mathrm{Mot}/\mathbb{F}}$  is a free Lie algebra in the category of pure motives. So (33) implies that the  $M^\vee$ -isotypical component of the space of generators is isomorphic to  $\mathrm{Ext}_{\mathrm{Mot}/\mathbb{F}}^1(\mathbb{Q}(0), M)$ .

Our next goal is to relate the motivic and Hodge correlators.

### The canonical period map.

**Lemma 1.10** *A choice of generators of the Lie algebra  $L_{\mathrm{Hod}}$  provides a period map*

$$p : \mathcal{L}ie_{\mathrm{Hod}} \longrightarrow i\mathbb{R}. \quad (36)$$

**Proof.** A choice of generators of the Lie algebra  $L_{\mathrm{Hod}}$  is just the same thing as a projection

$$\mathcal{L}ie_{\mathrm{Hod}} \longrightarrow \oplus_{[H]} \mathrm{Ext}_{\mathbb{R}\text{-MHS}}^1(\mathbb{R}(0), H) \otimes H^\vee, \quad (37)$$

where the sum is over the set of isomorphism classes of simple objects in the category of  $\mathbb{R}$ -Hodge structures. Next, for any simple object  $H$  there is a canonical map

$$p_{[H]} : \mathrm{Ext}_{\mathbb{R}\text{-MHS}}^1(\mathbb{R}(0), H) \otimes H^\vee \longrightarrow i\mathbb{R}. \quad (38)$$

Recall that given an  $\mathbb{R}$ -Hodge structure  $H$ , we have the formula

$$\mathrm{Ext}_{\mathbb{R}\text{-MHS}}^1(\mathbb{R}(0), H) = \mathrm{CoKer}\left(W_0 H \oplus F^0(W_0 H)_{\mathbb{C}} \longrightarrow (W_0 H)_{\mathbb{C}}\right). \quad (39)$$

Thus for a pure  $\mathbb{R}$ -Hodge structure  $H$  of Hodge degrees  $(-p, -q) + (-q, -p)$ , where  $p, q \geq 1$ , we have  $\mathrm{Ext}_{\mathbb{R}\text{-MHS}}^1(\mathbb{R}(0), H) = (H \otimes \mathbb{C})/H$ . So we arrive to a canonical pairing (38).

We define a period map (36) as the composition of the map (37) with the sum of the maps  $p_{[H]}$ . The Lemma is proved.

Therefore the canonical generators of the Lie algebra  $L_{\mathrm{Hod}}$ , discussed in Section 1.7, give rise to a canonical period map.

**Relating motivic and Hodge correlators.** Let  $X$  be a smooth complex curve. Just like in the case of the motivic correlators, the mixed  $\mathbb{R}$ -Hodge structure on  $L(X - S, v_0)$  plus the isomorphism (23) lead to the Hodge version of the motivic correlator map (35):

$$\text{Cor}_{\text{Hod}} : \mathcal{CLie}_{\mathcal{H}; X, S^*}^{\vee}(1) \longrightarrow \mathcal{Lie}_{\text{Hod}}. \quad (40)$$

Here  $\mathcal{CLie}_{\mathcal{H}; X, S^*}^{\vee}$  is the cyclic Lie coalgebra assigned to the  $\mathbb{R}$ -Hodge structure  $\text{gr}^W H^1(X - S; \mathbb{R})$ . Forgetting the Hodge bigrading, there is an isomorphism of Lie algebras over  $\mathbb{C}$

$$\mathcal{CLie}_{\mathcal{H}; X, S^*}^{\vee} \otimes \mathbb{C} \longrightarrow \mathcal{CLie}_{X, S^*}^{\vee}.$$

Using this, and combining (40) with the canonical period map (36) we arrive at a map

$$\mathcal{CLie}_{X, S^*}^{\vee}(1) \xrightarrow{\text{Cor}_{\text{Hod}}} \mathcal{Lie}_{\text{Hod}} \otimes \mathbb{C} \xrightarrow{p} \mathbb{C}. \quad (41)$$

The following result is an immediate corollary of Theorem 1.9 and the definitions.

**Theorem 1.11** *The composition (41) coincides with the Hodge correlator map.*

Let us assume now the motivic formalism. Then the  $\mathbb{R}$ -Hodge realization functor provides a homomorphism of Lie coalgebras

$$r_{\text{Hod}} : \mathcal{L}_{\text{Mot}/\mathbb{C}} \longrightarrow \mathcal{L}_{\text{Hod}}.$$

**Corollary 1.12** *Assuming the motivic formalism, the complexification of the composition*

$$\mathcal{CLie}_{\mathcal{M}; X, S^*}^{\vee}(1) \xrightarrow{\text{Cor}_{\text{Mot}}} \mathcal{L}_{\text{Mot}/\mathbb{C}} \xrightarrow{r_{\text{Hod}}} \mathcal{Lie}_{\text{Hod}} \xrightarrow{p} i\mathbb{R}.$$

*coincides with the Hodge correlator map.*

**Conclusion.** The source of the motivic correlator map (35) is an explicitly defined Lie coalgebra. Therefore motivic correlators come together with an explicit formula for their coproduct. The real period of the motivic correlator is the Hodge correlator. This, together with basic isomorphism (33), is all we need for the arithmetic analysis of the Hodge correlators.

**Example: Rankin-Selberg integrals and Beilinson's elements as correlators.** Here is how the story discussed in Section 1.2 fits in the correlator framework. Recall that  $M$  is a modular curve,  $\overline{M}$  its compactification,  $a, b$  degree zero cuspidal divisors, and  $g_a, g_b$  are invertible functions on  $M$  whose divisors are integral multiples of  $a$  and  $b$ . The Green function  $G(a, t)$  of the divisor  $a$  is an integral multiple of  $\log |g_a(t)|^2$ , and similarly for  $G(b, t)$ . Therefore integral (3) equals, up to a rational multiple, to the Hodge correlator integral provided by the cyclic word  $\mathcal{C}(\{a_0\} \otimes \{a_1\} \otimes f(z)dz)$ , see Fig 2:

$$\text{Cor}_{\mathcal{H}}(\{a_0\} \otimes \{a_1\} \otimes f(z)dz) = \int_{M(\mathbb{C})} G(a_0, t) d^{\mathbb{C}} G(a_1, t) \wedge f(z)dz. \quad (42)$$

The latter coincides with a Rankin-Selberg integral.

Let  $M_f$  be the dual to the pure weight two motive corresponding to the Hecke eigenform  $f(z)$ . It is a direct summand of the motive  $H_1(\overline{M})$ . The motivic correlator is an element

$$\text{Cor}_{\text{Mot}}\left((\{a_0\} \otimes \{a_1\} \otimes M_f^\vee)(1)\right) \in \mathcal{L}_{\text{Mot}}. \quad (43)$$

It lies in the  $M_f(1)^\vee$ -isotypical component of  $\mathcal{L}_{\text{Mot}}$ . Since  $a_i$  are torsion classes in the Jacobian of  $\overline{M}$ , we prove (Lemma 11.10) that (43) is in the kernel of the coproduct  $\delta$ . So, combining with the isomorphism (33), we get an element

$$\text{Cor}_{\text{Mot}}\left((\{a_0\} \otimes \{a_1\} \otimes M_f^\vee)(1)\right) \in \text{Ext}_{\text{Mot}/\mathbb{Q}}^1(\mathbb{Q}(0), M_f(1)) \otimes M_f(1)^\vee.$$

According to Beilinson's conjecture, one should have an isomorphism

$$\text{Ext}_{\text{Mot}/\mathbb{Q}}^1(\mathbb{Q}(0), H_1(\overline{M})(1)) = K_2(\overline{M}) \otimes \mathbb{Q}.$$

Using this isomorphism, the motivic correlator is related to Beilinson's element (1) by the formula

$$\text{Cor}_{\text{Mot}}\left((\{a_0\} \otimes \{a_1\} \otimes M_f^\vee)(1)\right) = \{g_a, g_b\} \otimes M_f(1)^\vee \in K_2(\overline{M}) \otimes M_f(1)^\vee.$$

By Corollary 1.12, the real period map on this element is given by the Rankin-Selberg integral.

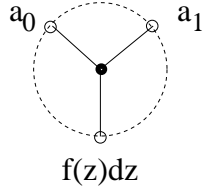


Figure 2: The Feynman diagram for Beilinson's element.

**A conjecture expressing  $L(\text{Sym}^2 M_f, 3)$  via Hodge correlators on modular curves.** Let us assume, for simplicity only, that the coefficients of the Hecke form  $f(z)$  are in  $\mathbb{Q}$ , i.e. the motive  $M_f$  is defined over  $\mathbb{Q}$ . Then Beilinson's conjecture predicts that

$$\dim_{\mathbb{Q}} \text{Ext}_{\text{Mot}/\mathbb{Q}}^1(\mathbb{Q}(0), \text{Sym}^2 M_f(1)) = 2.$$

**Conjecture 1.13** *The space  $\text{Ext}_{\text{Mot}/\mathbb{Q}}^1(\mathbb{Q}(0), \text{Sym}^2 M_f(1))$  is generated by the motivic correlators*

$$\text{Cor}_{\text{Mot}}\left((\{a_1\} \otimes \{a_2\} \otimes M_f^\vee \otimes M_f^\vee)(1) + (\{b_1\} \otimes M_f^\vee \otimes \{b_2\} \otimes M_f^\vee)(1)\right) \quad (44)$$

for certain degree zero cuspidal divisors  $a_i$  and  $b_i$  on the universal modular curve, which satisfy an explicit “coproduct zero” condition.

Precisely, an element (44) gives rise to an element of  $\text{Ext}_{\text{Mot}/\mathbb{Q}}^1(\mathbb{Q}(0), \text{Sym}^2 M_f(1))$  if and only if it is killed by the coproduct in the motivic Lie coalgebra. We elaborate in Example 2 of Section 11.3 the “coproduct zero” condition for any pair of cuspidal divisors  $a_i, b_i$  entering element (44). Finally, one should relate  $L(\text{Sym}^2 M_f, 3)$  to the determinant of the  $2 \times 2$  matrix, whose entries are given by the Hodge correlators of elements (44).

### 1.11 Motivic multiple $L$ -values

Here are some arithmetic applications obtained by picking certain specific pairs  $(X, S)$ , as well as certain specific motivic correlators.

The image  $\mathcal{G}(X - S, v_0)$  of the map (35) is called the *motivic Galois Lie algebra of  $X - S$* . In the  $l$ -adic setting it is the Lie algebra of the image of the Galois group  $\text{Gal}(\overline{F}/F(\mu_{l^\infty}))$  acting on the pro- $l$  completion  $\pi_1^{(l)}(X - S, v_0)$  of the fundamental group.

Specializing to the case when  $X$  is one of the following:

(i)  $\mathbb{G}_m$ , (ii) an elliptic CM curve, (iii) a Fermat curve, (iv) a modular curve, and choosing the subset  $S$  appropriately, we arrive at the definition of *motivic multiple  $L$ -values* corresponding to certain algebraic Hecke characters of  $\mathbb{Q}$ ,  $\mathbb{Q}(\sqrt{-d})$ ,  $\mathbb{Q}(\mu_N)$ , or weight two modular forms, respectively. They span a Galois Lie coalgebra, called the *multiple  $L$ -values Lie coalgebra*.

In the  $l$ -adic setting there is one more interesting case: (v) a Drinfeld modular curve.

Given triples  $(X', S', v'_0)$  and  $(X, S, v_0)$ , a map  $p : X' \rightarrow X$  such that  $p(S') = S$  and  $p(v'_0) = v_0$  induces a map of the motivic fundamental Lie algebras  $p : L(X' - S', v'_0) \rightarrow L(X - S, v_0)$ , and hence a map of the Galois Lie algebras  $p : \mathcal{G}(X' - S', v'_0) \rightarrow \mathcal{G}(X - S, v_0)$ , and the dual map of the Galois Lie coalgebras.

In each of the cases (i)-(v), the open curves  $X - S$  form a tower. For instance in the case (iv) it is the modular tower. Therefore the corresponding Galois Lie coalgebras form an inductive system. In many cases there is a canonical “averaged tangential base point” (Section 9.4), so below we ignore the base point. There is an adelic description of the limiting Galois Lie coalgebras. It is given in terms of the corresponding base field in the first three cases, and via the automorphic representations attached to the weight two modular forms in the modular curve case. In the latter case there is a generalization dealing with modular forms of arbitrary integral weight  $k \geq 2$ , which we will elaborate elsewhere.

Here is a description of the pairs  $(X, S)$  and the corresponding towers.

1. Let  $X = \mathbb{P}^1$ , and  $S$  is an arbitrary subset of  $\mathbb{P}^1$ . This is the situation studied in [G7], [G9].
  - a) A very interesting case is when  $S = \{0\} \cup \{\infty\} \cup \mu_N$ . It was studied in [G4], [DG].

*The tower.* The curves  $\mathbb{G}_m - \mu_N$ , parametrized by positive integers  $N$ , form a “tower” for the isogenies  $\mathbb{G}_m - \mu_{MN} \rightarrow \mathbb{G}_m - \mu_N$ .

2. The curve is an elliptic curve  $E$ , and  $S$  is any subset. An interesting special case is when  $S = E[N]$  is the subgroup of  $N$ -torsion points of  $E$ . *The tower* in this case is formed by the isogenies  $E - E[MN] \rightarrow E - E[N]$ .

There are the following ramifications:

- a) The universal elliptic curve  $\mathcal{E}$  with the level  $N$  structure over the modular curve.
  - s) The curve  $E$  is a CM curve,  $\mathcal{N}$  an ideal in the endomorphism ring  $\text{End}(E)$ , and  $S$  is the subgroup  $E[\mathcal{N}]$  of the  $\mathcal{N}$ -torsion points.
3. The curve is the Fermat curve  $\mathbb{F}_N$ , given in  $\mathbb{P}^2$  by the equation  $x^N + y^N = z^N$ , and  $S$  is the intersection of  $\mathbb{F}_N$  with the coordinate triangle  $x = 0, y = 0, z = 0$  in  $\mathbb{P}^2$ .

*The tower.* It is given by the maps  $\mathbb{F}_{MN} \rightarrow \mathbb{F}_N$ ,  $(x, y, z) \mapsto (x^M, y^M, z^M)$ .

4. The curve  $X = X(N)$  is the level  $N$  modular curve,  $S$  is the set of the cusps. There is a tower of modular curves, and especially interesting situation appears at the limit, i.e. when  $\mathcal{M} := \lim_{\leftarrow} X(N)$  is the universal modular curve and  $S$  is the set of its cusps.
5. Let  $A$  be the ring of rational functions on a regular projective curve over a finite field, which are regular at a chosen point  $\infty$  of the curve. Let  $I$  be an ideal in  $A$ . The Drinfeld modular curve is the moduli space of the rank 2 elliptic modules with the  $I$ -level structure. Take it as our curve  $X$ , and let  $S$  is the set of the cusps. There is a tower of the Drinfeld modular curves. In the limit we get the universal Drinfeld modular curve.

In each of these cases we get a supply of elements of the motivic Lie coalgebras. These are

1. “Cyclic” versions of motivic multiple polylogarithms.
  - a) Motivic multiple  $L$ -values for the Dirichlet characters of  $\mathbb{Q}$ .
2. Motivic multiple elliptic polylogarithms, and their restrictions to the torsion points.
  - a) Higher (motivic) analogs of modular units.
  - b) Motivic multiple  $L$ -values for Hecke Grössencharacters of imaginary quadratic fields.
3. Motivic multiple  $L$ -values for the Jacobi sums Grössencharacters of  $\mathbb{Q}(\mu_N)$ .
4. Motivic multiple  $L$ -values for the modular forms of weight 2.
5. Similar objects in the function field case.

We prove in Section 11 that the real periods of the motivic multiple elliptic polylogarithms are the generalized Eisenstein-Kronecker series, defined in [G1] as integrals of the classical Eisenstein-Kronecker series, and obtain similar results in the rational case.

**What are the multiple  $L$ -values?** In each of the above cases the motivic multiple  $L$ -values are certain natural collections of elements of the motivic Lie coalgebra, closed under the Lie cobracket. Their periods are numbers, called the multiple  $L$ -values.

The “classical” motivic  $L$ -values are the elements killed by the Lie cobracket. Moreover, they are precisely the cogenerators of the corresponding motivic Lie coalgebra, providing the name for the latter. Their periods are special values of  $L$ -functions.

The “classical” motivic  $L$ -values are motivic cohomology classes. In general the motivic multiple  $L$ -values are no longer motivic cohomology classes.

**Example.** The first multiple  $L$ -values discovered were Euler’s multiple  $\zeta$ -numbers

$$\zeta(n_1, \dots, n_m) = \sum_{0 < k_1 < \dots < k_m} \frac{1}{k_1^{n_1} \dots k_m^{n_m}}.$$

The motivic  $\zeta$ -values are the elements  $\zeta_{\mathcal{M}}(2n-1) \in \text{Ext}_{\text{Mot}/\mathbb{Z}}^1(\mathbb{Q}(0), \mathbb{Q}(n))$ . The latter Ext-group is one-dimensional, and is generated by  $\zeta_{\mathcal{M}}(2n-1)$ . The motivic multiple zeta values  $\zeta_{\mathcal{M}}(n_1, \dots, n_m)$  are the elements of the motivic Lie coalgebra of the category of mixed Tate motives over  $\text{Spec}(\mathbb{Z})$ . Conjecturally they span the latter.

Similarly, each motivic multiple  $L$ -values Lie coalgebra is tied up to a certain subcategory of the category of mixed motives. A very interesting question is how far it is from the motivic Lie coalgebra which governs that mixed category. See [G4] for the case 1a) and [G10] for the case 2b), where the gap in each case was related to the geometry of certain modular varieties.

In all above cases the motivic cohomology group responsible for the  $L$ -values were one-dimensional.

**Question.** Can one define multiple  $L$ -values in a more general situation?

### 1.12 Coda: Feynman integrals, motivic correlators and the correspondence principle

We suggested in Section 8 of [G7] the following picture relating Feynman integrals and mixed motives. If correlators of a Feynman integral are periods, they should be upgraded to motivic correlators, which lie in the motivic Lie coalgebra, and whose periods are the Feynman correlators. Moreover, the obtained motivic correlators should be closed under the coproduct in the motivic Lie coalgebra (otherwise the original set of Feynman correlators was not complete). Finally, according to the correspondence principle in loc. cit. the coproduct of the motivic correlators should be calculated as follows. There should be a combinatorially defined Lie coalgebra – the renormalization Lie coalgebra – and the motivic correlators should provide a homomorphism

$$\text{the renormalization Lie coalgebra} \longrightarrow \text{the motivic Lie coalgebra.} \quad (45)$$

This paper gives an example of realization of this program, including the correspondence principle. Indeed, the Hodge correlators are the tree level correlators of the Feynman integral from Section 1.6. We upgrade them to motivic correlators. They span a Lie coalgebra, which is nothing else but the Galois Lie coalgebra  $\mathcal{G}(X - S, v_0)$ . The Lie coalgebra  $\mathcal{CLie}_{X,S^*}^\vee$  plays the role of the renormalization Lie coalgebra: There is a canonical Lie coalgebra homomorphism  $\mathcal{CLie}_{X,S^*}^\vee \rightarrow \mathcal{G}(X - S, v_0)$  describing the coproduct of motivic correlators, which we think of as the homomorphism (45).

However I do not know a renormalization theory interpretation the Lie coalgebra  $\mathcal{CLie}_{X,S^*}^\vee$ .

**Problem.** Develop a similar picture for Feynman integrals whose correlators are periods.

### 1.13 The structure of the paper

Hodge correlators are defined in Section 2. We show in Section 12 that they indeed are correlators of a Feynman integral. However we do not use this anywhere else in the paper.

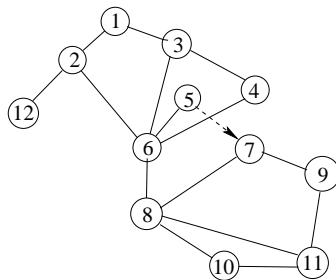


Figure 3: Leitfaden.

In Section 3 we define the Hodge complex of a variation of pure Hodge structures. We introduce a DGA structure on the direct sum of Hodge complexes over isomorphism classes of pure variations on a complex projective variety  $X$  – we call it the Hodge DGA of  $X$ . We define a twistor transform and show that it embeds the Hodge DGA of  $X$  into the De Rham DGA of  $X \times \mathbb{R}$ . This clarifies tremendously the formulas used to define the Hodge DGA.

Based on results of Section 3, in Section 4 we encode variations of real mixed Hodge structures by Green data satisfying certain non-linear quadratic Maurer-Cartan differential equations.

In Section 5 we introduce several Lie/DG Lie coalgebras related to plane trees/forests decorated by  $\mathrm{gr}^W H^1(X - S)$ . They generalize the ones defined in [G4] and [G9] in the case  $X = \mathbb{P}^1$ .

In Section 6 we derive differential equations for the Hodge correlators for families of curves. We show that the Hodge correlators lie naturally in a DG Lie algebra obtained by combining constructions of Sections 3 and 5, and satisfy there the Maurer-Cartan differential equations.

In Section 7 we describe the Lie algebra of special derivations of  $\mathrm{gr}^W \pi_1^{\mathrm{nil}}(X - S, v_0)$ . We show that it is the Lie algebra  $\mathcal{CLie}_{X, S^*}$ . Since the Hodge correlator  $\mathbb{G}$  lies in  $\mathcal{CLie}_{X, S^*}$ , we view it as a special derivation of the Lie algebra  $\mathrm{gr}^W \pi_1^{\mathrm{nil}}(X - S, v_0)$ . So by the construction of Section 4 the Hodge correlator  $\mathbb{G}$  provides a real mixed Hodge structure on  $\pi_1^{\mathrm{nil}}(X - S, v_0)$ . In Section 8 we prove that it is the standard one.

In Sections 9 we translate results of Section 7 into the motivic framework. Using this we introduce *motivic correlators*. They are canonical elements in the motivic Lie coalgebra. The Hodge correlators are the real periods of the motivic correlators.

In Section 10 we consider examples. We show how the classical and elliptic polylogarithms appear as Hodge correlators for simple Feynman diagrams. We prove that for an elliptic curve the Hodge correlators can be expressed by the multiple Eisenstein-Kronecker series.

In Section 11 we show that Hodge / motivic correlators on modular curves generalize the Rankin-Selberg integrals / Beilinson's elements in  $K_2$  of modular curves. Going to the limit in the tower of modular curves we get an automorphic adelic description of the correlators on modular curves.

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## 2 Hodge correlators on curves

### 2.1 The form $\omega_m(\varphi_0, \dots, \varphi_m)$ and its properties

Let  $\varphi_0, \dots, \varphi_m$  be any  $m + 1$  smooth forms on a complex manifold  $M$ . Let us define a form  $\omega_m(\varphi_0, \dots, \varphi_m)$ . First, given elements  $f_i$  of degrees  $|f_i|$  and a function  $F(f_1, \dots, f_m)$  set

$$\mathrm{Alt}_m F(f_1, \dots, f_m) := \sum_{\sigma \in \Sigma_m} \mathrm{sgn}_{\sigma; f_1, \dots, f_m} F(f_{\sigma(1)}, \dots, f_{\sigma(m)}), \quad (46)$$



where the sign of the transposition of  $f_1, f_2$  is  $(-1)^{(|f_1|+1)(|f_2|+1)}$ , and the sign of a permutation written as a product of transpositions is the product of the signs of the transpositions. It does not depend on decomposition of the permutation into a product of transpositions. Set

$$\omega_m(\varphi_0, \dots, \varphi_m) := \frac{1}{(m+1)!} \text{Alt}_{m+1} \left( \sum_{k=0}^m (-1)^k \varphi_0 \wedge \partial \varphi_1 \wedge \dots \wedge \partial \varphi_k \wedge \bar{\partial} \varphi_{k+1} \wedge \dots \wedge \bar{\partial} \varphi_m \right). \quad (47)$$

Here we apply the alternation sign convention to a function of the forms  $\varphi_0, \dots, \varphi_m$ . So if they are 0-forms, we use the standard alternation.

Denote by  $\mathcal{A}_M^k$  the space of smooth  $k$ -forms on  $M$ . There is a degree  $m$  linear map

$$\omega_m : \Lambda^m \mathcal{A}_M^\bullet \longrightarrow \mathcal{A}_M^\bullet, \quad \omega_m : \varphi_0 \wedge \dots \wedge \varphi_m \longmapsto \omega_m(\varphi_0, \dots, \varphi_m).$$

For example  $\omega_0(\varphi_0) = \varphi_0$ ,

$$\omega_1(\varphi_0 \wedge \varphi_1) = \frac{1}{2} \left( \varphi_0 \wedge \bar{\partial} \varphi_1 - \varphi_0 \wedge \partial \varphi_1 + (-1)^{(|\varphi_0|+1)(|\varphi_1|+1)} (\varphi_1 \wedge \bar{\partial} \varphi_0 - \varphi_1 \wedge \partial \varphi_0) \right).$$

The first key property of this map is the following:

**Lemma 2.1**

$$\begin{aligned} d\omega_m(\varphi_0, \dots, \varphi_m) &= (-1)^m \partial \varphi_0 \wedge \dots \wedge \partial \varphi_m + \bar{\partial} \varphi_0 \wedge \dots \wedge \bar{\partial} \varphi_m + \\ &\quad \frac{1}{m!} \text{Alt}_{m+1} \left( (-1)^{|\varphi_0|} \bar{\partial} \partial \varphi_0 \wedge \omega_{m-1}(\varphi_1, \dots, \varphi_m) \right). \end{aligned} \quad (48)$$

(The last term can be written as  $\sum_{j=0}^m (-1)^j \pm \bar{\partial} \partial \varphi_j \wedge \omega_{m-1}(\varphi_0, \dots, \widehat{\varphi}_j, \dots, \varphi_m)$ ).

**Proof.** There are the following three identities, which allow to move differential operators under the alternation sign, without moving the forms  $\varphi_i$ :

$$\text{Alt}_2(\bar{\partial} \varphi_1 \wedge \partial \varphi_2) = \text{Alt}_2(\partial \varphi_1 \wedge \bar{\partial} \varphi_2). \quad (49)$$

$$\text{Alt}_2 \left( (-1)^{|\varphi_1|} \varphi_1 \wedge \partial \bar{\partial} \varphi_2 \right) = -\text{Alt}_2 \left( (-1)^{|\varphi_1|} \partial \bar{\partial} \varphi_1 \wedge \varphi_2 \right). \quad (50)$$

$$\text{Alt}_2 \left( \partial \varphi_1 \wedge \bar{\partial} \partial \varphi_2 \right) = \text{Alt}_2 \left( (-1)^{|\varphi_1|+1} \bar{\partial} \partial \varphi_1 \wedge \partial \varphi_2 \right). \quad (51)$$

Indeed, identity (49) is proved as follows:

$$\text{Alt}_2(\bar{\partial} \varphi_1 \wedge \partial \varphi_2) = (-1)^{(|\varphi_1|+1)(|\varphi_2|+1)} \text{Alt}_2(\bar{\partial} \varphi_2 \wedge \partial \varphi_1) = \text{Alt}_2(\partial \varphi_1 \wedge \bar{\partial} \varphi_2).$$

Identity (50) is proved by

$$\begin{aligned} \text{Alt}_2 \left( (-1)^{|\varphi_1|} \varphi_1 \wedge \partial \bar{\partial} \varphi_2 \right) &= \text{Alt}_2 \left( (-1)^{|\varphi_1|+|\varphi_1||\varphi_2|} \partial \bar{\partial} \varphi_2 \wedge \varphi_1 \right) \\ &= \text{Alt}_2 \left( (-1)^{|\varphi_2|+|\varphi_2||\varphi_1|+(|\varphi_1|+1)(|\varphi_2|+1)} \partial \bar{\partial} \varphi_1 \wedge \varphi_2 \right) = -\text{Alt}_2 \left( (-1)^{|\varphi_1|} \partial \bar{\partial} \varphi_1 \wedge \varphi_2 \right). \end{aligned}$$

The third identity is checked in a similar way.

One has

$$\begin{aligned}
d\omega_m(\varphi_0, \dots, \varphi_m) = & \\
& \frac{1}{(m+1)!} \text{Alt}_{m+1} \left( \sum_{k=0}^m (-1)^k \cdot (\partial\varphi_0 + \bar{\partial}\varphi_0) \wedge \partial\varphi_1 \wedge \dots \wedge \partial\varphi_k \wedge \bar{\partial}\varphi_{k+1} \wedge \dots \wedge \bar{\partial}\varphi_m \right. \\
& + \sum_{k=1}^m (-1)^{k+|\varphi_0|} k \cdot \varphi_0 \wedge \bar{\partial}\partial\varphi_1 \wedge \partial\varphi_2 \wedge \dots \wedge \partial\varphi_k \wedge \bar{\partial}\varphi_{k+1} \wedge \dots \wedge \bar{\partial}\varphi_m \\
& \left. + \sum_{k=0}^{m-1} (-1)^{|\varphi_0|+\dots+|\varphi_k|} (m-k) \cdot \varphi_0 \wedge \partial\varphi_1 \wedge \dots \wedge \partial\varphi_k \wedge \partial\bar{\partial}\varphi_{k+1} \wedge \bar{\partial}\varphi_{k+2} \wedge \dots \wedge \bar{\partial}\varphi_m \right).
\end{aligned}$$

Using (49), the first term contributes the first term in the right hand side of (48).

Using (51), the third term is written as

$$\frac{1}{(m+1)!} \text{Alt}_{m+1} \left( \sum_{k=0}^{m-1} (m-k) (-1)^{k+|\varphi_0|} \cdot \varphi_0 \wedge \partial\bar{\partial}\varphi_1 \wedge \partial\varphi_2 \wedge \dots \wedge \partial\varphi_{k+1} \wedge \bar{\partial}\varphi_{k+2} \wedge \dots \wedge \bar{\partial}\varphi_m \right).$$

Using (51) and  $\partial\bar{\partial} = -\bar{\partial}\partial$ , the last two terms in the expression for  $d\omega_m(\varphi_0, \dots, \varphi_m)$  give

$$\begin{aligned}
& \frac{1}{(m+1)!} \text{Alt}_{m+1} \left( \sum_{k=1}^m (-1)^{k-1+|\varphi_0|} k \cdot \bar{\partial}\partial\varphi_0 \wedge \varphi_1 \wedge \partial\varphi_2 \wedge \dots \wedge \partial\varphi_k \wedge \bar{\partial}\varphi_{k+1} \wedge \dots \wedge \bar{\partial}\varphi_m \right. \\
& \left. + \sum_{k=0}^{m-1} (-1)^{k+|\varphi_0|} (m-k) \cdot \bar{\partial}\partial\varphi_0 \wedge \varphi_1 \wedge \partial\varphi_2 \wedge \dots \wedge \partial\varphi_{k+1} \wedge \bar{\partial}\varphi_{k+2} \wedge \dots \wedge \bar{\partial}\varphi_m \right).
\end{aligned}$$

Changing the summation in the second term from  $0 \leq k \leq m-1$  to  $1 \leq k \leq m$ , and using  $k + (m+1-k) = m+1$ , we see that it coincides with the term containing Laplacians in (48). The lemma is proved.

**Remark.** Let  $f_i$  be rational functions on a complex algebraic variety. Let  $\varphi_i := \log |f_i|$ . Then the form (47) is a part of a cocycle representing the product in real Deligne cohomology of 1-cocycles  $(\log |f_i|, d \log f_i)$  representing classes in  $H_{\mathcal{D}}^1(X, \mathbb{R}(1))$ , see Proposition 2.1 of [GZ]. The form (47) in this set up was used for the definition of Chow polylogarithms and, more generally, in the construction of the canonical regulator map on motivic complexes ([G2], [G8]).

Recall  $d^{\mathbb{C}} := \partial - \bar{\partial}$ . Let us introduce the following differential forms:

$$\xi_m(\varphi_0, \dots, \varphi_m) := \frac{1}{(m+1)!} \text{Alt}_{m+1} \left( \varphi_0 \wedge d^{\mathbb{C}}\varphi_1 \wedge \dots \wedge d^{\mathbb{C}}\varphi_m \right) = \quad (52)$$

$$\frac{1}{(m+1)!} \text{Alt}_{m+1} \left( \sum_{k=0}^m (-1)^k \binom{m}{k} \varphi_0 \wedge \partial\varphi_1 \wedge \dots \wedge \partial\varphi_k \wedge \bar{\partial}\varphi_{k+1} \wedge \dots \wedge \bar{\partial}\varphi_m \right).$$

$$\eta_{m+1}(\varphi_0, \dots, \varphi_m) := \frac{1}{(m+1)!} \text{Alt}_{m+1} \left( d^{\mathbb{C}}\varphi_0 \wedge \dots \wedge d^{\mathbb{C}}\varphi_m \right) = \quad (53)$$

$$\frac{1}{(m+1)!} \text{Alt}_{m+1} \left( \sum_{k=0}^m (-1)^k \binom{m+1}{k+1} \partial\varphi_0 \wedge \dots \wedge \partial\varphi_k \wedge \bar{\partial}\varphi_{k+1} \wedge \dots \wedge \bar{\partial}\varphi_m \right).$$

Every summand in the form  $\eta$  appears with the coefficient  $\pm 1$ . Since  $(d^{\mathbb{C}})^2 = 0$ , we have

$$d^{\mathbb{C}}\xi_m(\varphi_0, \dots, \varphi_m) = \eta_m(\varphi_0, \dots, \varphi_m). \quad (54)$$

**Example.** Assume for simplicity that  $|\varphi_i| = 0$ . Then  $\xi_1(\varphi_0, \varphi_1) = \omega_1(\varphi_0, \varphi_1)$ ,

## 2.2 Green functions on a Riemann surface

Let  $X$  be a smooth compact complex algebraic curve of genus  $g$ . There is a canonical hermitian structure on the space  $\Omega^1(X)$  of holomorphic differentials on  $X$  given by

$$\langle \alpha_k, \alpha_l \rangle := \frac{i}{2} \int_{X(\mathbb{C})} \alpha_k \wedge \bar{\alpha}_l. \quad (55)$$

Let  $\alpha_1, \dots, \alpha_g$  be an orthonormal basis with respect to this form.

Let  $\mu$  be a 2-current on  $X(\mathbb{C})$  of volume one:  $\int_{X(\mathbb{C})} \mu = 1$ . The corresponding Green current  $G_\mu(x, y)$  is a 0-current on  $X(\mathbb{C}) \times X(\mathbb{C})$  satisfying a differential equation

$$\frac{1}{2\pi i} \bar{\partial} \partial G_\mu(x, y) = \delta_{\Delta_X} - \left( p_1^* \mu + p_2^* \mu - \frac{i}{2} \sum_{k=1}^g (p_1^* \alpha_k \wedge p_2^* \bar{\alpha}_k + p_2^* \alpha_k \wedge p_1^* \bar{\alpha}_k) \right). \quad (56)$$

There exists a unique up to a constant solution of this equation. Indeed, the kernel of the Laplacian  $\bar{\partial} \partial$  on  $X(\mathbb{C}) \times X(\mathbb{C})$  consists of constants. One has a symmetry relation:

$$G_\mu(x, y) = G_\mu(y, x).$$

Indeed, the right hand side of (56) is invariant under the permutation of the factors in  $X \times X$ . So the difference is a constant, and vanishes at the diagonal.

A point  $a \in X(\mathbb{C})$  provides a volume one 2-current  $\delta_a$  on  $X(\mathbb{C})$ . So letting  $\mu := \delta_a$ , we get a Green function denoted  $G_a(x, y)$ . Then the current on the right hand side of (56) is smooth outside of the divisor  $\Delta \cup (\{a\} \times X) \cup (X \times \{a\})$ . Since the Laplacian is an elliptic operator, the restriction of the Green current to the complement of this divisor is represented by a smooth function, called the Green function, also denoted by  $G_a(x, y)$ . The Green function  $G_a(x, y)$  near the singularity divisor looks like  $\log r$ , where  $r$  is a distance to the divisor, plus a smooth function. So it has an integrable singularity, and thus provides a current on  $X(\mathbb{C}) \times X(\mathbb{C})$ , which coincides with the Green current.

**The Arakelov Green function.** A classical choice of  $\mu$  is given by a smooth volume form of total mass 1 on  $X(\mathbb{C})$ . In particular a metric on  $X(\mathbb{C})$  provides such a volume form. There is a canonical *Arakelov volume form* on  $X(\mathbb{C})$ :

$$\text{vol}_X := \frac{i}{2g} \sum_{\alpha=1}^g \omega_\alpha \wedge \bar{\omega}_\alpha, \quad \int_{X(\mathbb{C})} \text{vol}_X = 1.$$

The corresponding Green function is denoted by  $G_{\text{Ar}}(x, y)$ . It can be normalized so that

$$p_{2*}(G_{\text{Ar}}(x, y) p_1^* \text{vol}_X) = p_{1*}(G_{\text{Ar}}(x, y) p_2^* \text{vol}_X) = 0. \quad (57)$$

The Green function  $G_a(x, y)$  is expressed via the Arakelov Green function:

$$G_a(x, y) = G_{\text{Ar}}(x, y) - G_{\text{Ar}}(a, y) - G_{\text{Ar}}(x, a) + C. \quad (58)$$

Indeed, the right hand side satisfies differential equation (56).

**A normalization of  $G_a(x, y)$ .** Let us choose a non-zero tangent vector  $v$  at  $a$ . Let  $t$  be a local parameter at  $a$  such that  $dt(v) = 1$ . Then there exists a unique solution  $G_v(x, y)$  of (56) with  $\mu = \delta_a$  such that  $G_v(x, y) - \log |t|$  vanishes at  $x = a$ . Indeed, there is a unique normalized Arakelov Green function  $G_{\text{Ar},v}(x, y)$  such that  $G_{\text{Ar},v}(x, a) - \log |t|$  vanishes at  $x = a$ . Thanks to the symmetry, the same is true for  $G_{\text{Ar}}(a, y) - \log |t|$  vanishes at  $y = a$ . It remains to use (58) with  $C = 0$  there.

**Specialization.** Let  $F$  be a function defined in a punctured neighborhood of a point  $a$  on a complex curve  $X$ . Let  $v$  be a tangent vector at  $a$ . Choose a local parameter  $t$  in the neighborhood of  $a$  such that  $dt(v) = 1$ . Suppose that the function  $F(t)$  admits an asymptotic expansion

$$F(t) = \sum_n F_n(t) \log^n |\varepsilon|, \quad n \geq 0.$$

where  $F_n(t)$  are smooth functions near  $t = 0$ . We define the specialization  $\text{Sp}_v^{t \rightarrow a} F(t) := F_0(0)$ . We skip sometimes the superscript  $t \rightarrow 0$ . It follows immediately from the definitions that

$$\text{Sp}_v^{x \rightarrow a} G_v(x, y) = \text{Sp}_v^{y \rightarrow a} G_v(x, y) = 0. \quad (59)$$

We extend the Green function by linearity to the group of divisors on  $X(\mathbb{C})$ .

**Lemma 2.2** *If  $D_0$  is a degree zero divisor, then  $G_\mu(D_0, y) - G_{\mu'}(D_0, y)$  is a constant.*

**Proof.** One has  $\frac{1}{2\pi i} \bar{\partial} \partial (G_\mu(D_0, y) - G_{\mu'}(D_0, y)) = p_1^*(\mu - \mu') + p_2^*(\mu - \mu')$ . Thus the difference is  $p_1^*f + p_2^*f$  for a 0-current  $f$  on  $X(\mathbb{C})$ . The lemma follows.

**A 1-form  $\nu$ .** Let  $p : X \rightarrow B$  be a smooth family of complex projective algebraic varieties over a base  $B$ . Then  $\mathcal{H} := R_1 p_*(\mathbb{R})$  is a local system on  $B$  with the Gauss-Manin connection  $\nabla_{GM}$  and the fiber  $H_1(X_t, \mathbb{R})$  over  $t \in B$ . By the very definition  $H_1(X_t, \mathbb{Z})$  is flat for the Gauss-Manin connection.

The Albanese variety  $A_{X_t}$  of  $X_t$  is identified as a topological space with  $H_1(X_t, \mathbb{R})/H_1(X_t, \mathbb{Z})$ . The fibers of the local system  $\mathcal{H}$  can be viewed as the tangent spaces to Albanese varieties  $A_{X_t}$ .

Let  $\pi : X \times_B X \rightarrow B$  be the fibered product of  $X$  over  $B$ . Then there is a  $C^\infty$  1-form

$$\nu \in \mathcal{A}_{X \times_B X}^1 \otimes \pi^* \mathcal{H} \quad (60)$$

on  $X \times_B X$  with values in the local system  $\pi^* \mathcal{H}$ . Its value at a tangent vector  $(v_1, v_2)$  at  $(x_1, x_2)$  is obtained as follows. Let  $(x_1(t), x_2(t))$  be a smooth path from  $(x_1, x_2)$  in the direction  $(v_1, v_2)$ . Assuming that the points  $(x_1(t), x_2(t))$  are in the same fiber  $X_t$ , we get a section  $(x_1(t) - x_2(t)) \in A_{X_t}$ . The Gauss-Manin connection provides another section of the Albanese fibration passing through  $(x_1(0) - x_2(0))$ . We define  $\nu(v_1, v_2)$  to be the difference of the tangents to these sections with given velocity  $(v_1, v_2)$ .

**Green functions for a family of curves.** Let  $\mathcal{H}_{\mathbb{C}} := \mathcal{H} \otimes \mathbb{C} = \mathcal{H}_{\mathbb{C}}^{-1,0} \oplus \mathcal{H}_{\mathbb{C}}^{0,-1}$  be the Hodge decomposition. The intersection pairing  $H_1(X_t, \mathbb{Z}) \wedge H_1(X_t, \mathbb{Z}) \rightarrow \mathbb{Z}(1) := 2\pi i \mathbb{Z}$  provides a pairing

$$\langle *, * \rangle : \mathcal{H}_{\mathbb{C}}^{-1,0} \otimes \mathcal{H}_{\mathbb{C}}^{0,-1} \rightarrow \mathbb{C}.$$

The wedge product  $\nu \wedge \nu$ , followed by the pairing  $\langle *, * \rangle$ , provides a canonical  $(1, 1)$ -form

$$\langle \nu \wedge \nu \rangle \in \mathcal{A}_{X \times_B X}^{1,1}.$$

Denote by  $\langle \nu \wedge \nu \rangle^{[1 \times 1]}$  its  $1 \times 1$ -component on  $X \times_B X$ . Let  $\delta_\Delta$  be the  $\delta$ -function of the relative diagonal  $\Delta \subset X \times_B X$ . A section  $a : B \rightarrow X$  provides a divisor  $S_a \subset X \times_B X$ , whose fiber at  $t \in B$  is  $a_t \times X_t \cup X_t \times a_t$ . It gives rise to a  $(1, 1)$ -current  $\delta_{S_a}$  on  $X(\mathbb{C})$ .

**Definition 2.3** *Let  $p : X \rightarrow B$  be a proper map and  $a : B \rightarrow X$  is its section. A Green current  $G_a(x, y)$  is a 0-current on  $X(\mathbb{C}) \times_B X(\mathbb{C})$  satisfying the equation*

$$(2\pi i)^{-1} \bar{\partial} \partial G_a(x, y) = \delta_\Delta - \delta_{S_a} - \langle \nu \wedge \nu \rangle^{[1 \times 1]}. \quad (61)$$

The Green function exists fiberwise by the  $\bar{\partial} \partial$ -lemma. It determines a 0-current  $G_a(x, y)$  uniquely up to a function lifted from the base  $B(\mathbb{C})$ . Let us choose a non vanishing  $v \in T_a X$ . It provides a uniquely defined normalized Green function  $G_v(x, y)$ .

### 2.3 Hodge correlators for curves

**Decorated plane trivalent trees.** Let us recall some terminology. A *tree* is a connected graph without loops. The *external vertices* of a tree are the ones of valency 1. The rest of the vertices are *internal vertices*. The edges of a tree consist of *external edges*, i.e. the ones containing the external vertices, and *internal edges*. A *trivalent tree* is a tree whose internal vertices are of valency 3. We allow a trivalent tree without internal vertices. It is just one edge with two (external) vertices. A trivalent tree with  $m + 1$  external vertices has  $2m - 1$  edges. A *plane tree* is a tree without self intersections located on the plane.

**Definition 2.4** *A decoration of a tree  $T$  by elements of a set  $R$  is a map  $\{\text{external vertices of } T\} \rightarrow R$ .*

An example of a decorated plane trivalent tree is given on Figure 4.

Recall (Section 1.3) that  $S^* \subset X(\mathbb{C})$  and

$$V_{X, S^*}^\vee = \mathbb{C}[S^*] \oplus (\Omega_X^1 \oplus \bar{\Omega}_X^1).$$

Further,  $\mathcal{C}_{X, S^*}^\vee$  is the cyclic envelope of the tensor algebra of  $V_{X, S^*}^\vee$ . Its elements are cyclic words in  $V_{X, S^*}^\vee$ : for instance, one has  $\mathcal{C}(abc) = \mathcal{C}(cab)$ .

A 2-current  $\mu$  is *admissible* if it is either a smooth volume form of total mass 1 on  $X(\mathbb{C})$ , or the  $\delta$ -current  $\delta_a$  for some point  $a \in X(\mathbb{C})$ . Given an admissible 2-current  $\mu$ , we are going to define a linear map, the *Hodge correlator map*:

$$\text{Cor}_{\mathcal{H}, \mu} : \mathcal{C}_{X, S^*}^\vee \longrightarrow \mathbb{C}.$$

To define it, take a degree  $m + 1$  decomposable cyclic element

$$W = \mathcal{C}(v_0 \otimes v_1 \otimes \dots \otimes v_m), \quad v_i \in V_{X, S^*}^\vee.$$

The external vertices of a plane tree have a cyclic order provided by a chosen (say, clockwise) orientation of the plane. We say that a plane trivalent tree  $T$  is decorated by  $W$  if the external

vertices of the tree are decorated by the elements  $v_0, \dots, v_m$ , respecting the cyclic order of the vertices, as on Figure 4. Although the decoration depends on the presentation of  $W$  as a cyclic tensor product of the vectors  $v_i$ , constructions below depend only on  $W$  and not on such presentation. We are going to assign to  $W$  a  $2m$ -current  $\kappa_W$  on

$$X(\mathbb{C})\{\text{vertices of } T\}. \quad (62)$$

We assume that each of the vectors  $v_i$  in  $W$  is either a generator  $\{s_i\} \in \mathbb{C}[S^*]$ , or a 1-form  $\omega$

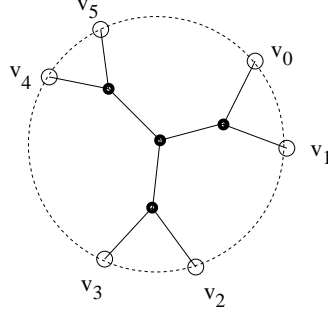


Figure 4: A decorated plane trivalent tree.

on  $X(\mathbb{C})$ . In Section 2.2 we assume in addition that if  $T$  has just one edge, both its vertices are  $S$ -decorated. Then a decoration of  $T$  provides a decomposition of the set of external edges into the  $S^*$ -decorated and *special* edges: the latter are decorated by 1-forms, the former by points of  $S^*$ .

Given an edge  $E$  of  $T$ , we assign to it a current  $\tilde{G}_E$  on  $X(\mathbb{C}) \times X(\mathbb{C})$ :

$$\tilde{G}_E := \begin{cases} \delta_X \omega & \text{if } E \text{ is a special external edge,} \\ G_\mu(x_1, x_2) & \text{otherwise.} \end{cases} \quad (63)$$

For each edge  $E$  of  $T$ , there is a projection of the set of vertices of  $T$  onto the set of vertices of  $E$ . It gives rise to a canonical projection

$$p_E : X(\mathbb{C})\{\text{vertices of } T\} \longrightarrow X(\mathbb{C})\{\text{vertices of } E\} = X(\mathbb{C}) \times X(\mathbb{C}).$$

We set

$$G_E := p_E^* \tilde{G}_E.$$

Since the map  $p_E$  is transversal to the wave front of the current  $\tilde{G}_E$ , the inverse image is well defined.

Given a finite set  $\mathcal{X}$ , there is a  $\mathbb{Z}/2\mathbb{Z}$ -torsor  $\text{or}_{\mathcal{X}}$ , called the *orientation torsor* of  $\mathcal{X}$ . Its elements are expressions  $x_1 \wedge \dots \wedge x_{|\mathcal{X}|}$ , where  $\{x_1, \dots, x_{|\mathcal{X}|}\} = \mathcal{X}$ ; interchanging two neighbors we change the sign of the expression. The *orientation torsor*  $\text{or}_T$  of a graph  $T$  is the orientation torsor of the set of edges of  $T$ . An orientation of the plane induces an orientation of a plane trivalent tree: Indeed, the orientation of the plane provides orientations of links of each of the vertices.

Let us choose an element

$$(E_0 \wedge \dots \wedge E_r) \wedge (E_{r+1} \wedge \dots \wedge E_{2m}) \in \text{or}_T$$

of the orientation torsor of our tree  $T$ , such that the edges  $E_0, \dots, E_r$  are internal or  $S$ -decorated, and the others are decorated by 1-forms. Set

$$\kappa_T(W) := \text{sgn}(E_0 \wedge \dots \wedge E_{2m}) \omega_r(G_{E_0} \wedge \dots \wedge G_{E_r}) \wedge G_{E_{r+1}} \wedge \dots \wedge G_{E_{2m}}, \quad (64)$$

where  $\text{sign}(E_0 \wedge \dots \wedge E_{2m}) \in \{\pm 1\}$  is the difference between the element  $E_0 \wedge \dots \wedge E_{2m}$  and the canonical generator of  $\text{or}_T$ .

**Lemma 2.5**  $\kappa_T(W)$  is a  $2m$ -current on (62).

**Proof.** Consider the smooth differential  $r$ -form

$$\omega_r(G_{E_0} \wedge \dots \wedge G_{E_r}) \quad (65)$$

on the complement to the diagonals in (62). We claim that it has integrable singularities, i.e. provides an  $r$ -current on (62). Indeed, the Green function is smooth outside of the diagonal. Its singularity near the diagonal is of type  $\log |t|$ , where  $t$  is a local equation of the diagonal i.e.  $G(x, y) - \log |t|$  is smooth near the diagonal. Since  $\omega_m$  is a linear map, it is sufficient to prove the claim for a specific collection of functions with such singularities. Since the problem is local, we can choose local equations  $f_i = 0$  of all the diagonals involved, and consider the functions  $\log |f_i|$  as such a specific collection. Then the claim follows from Theorem 2.4 in [G8]. The lemma follows.

Consider the projection map

$$p_T : X(\mathbb{C})^{\{\text{vertices of } T\}} \longrightarrow X(\mathbb{C})^{\{S^*\text{-decorated vertices of } T\}}. \quad (66)$$

**Definition 2.6** Let  $\mu$  be an admissible 2-current on  $X(\mathbb{C})$ , and  $W \in \mathcal{C}_{X, S^*}^\vee$ . Then the correlator  $\text{Cor}_{\mathcal{H}}(W)$  is a 0-current on the right hand side of (66), given by

$$\text{Cor}_{\mathcal{H}}(W) := p_{T*} \left( \sum_T \kappa_T(W) \right),$$

where the sum is over all plane trivalent trees  $T$  decorated by the cyclic word  $W$ .

The restriction of the current  $\text{Cor}_{\mathcal{H}}(W)$  to the complement of the diagonals is a smooth function.

**Remarks.** 1. The direct image  $p_{T*}$  is given by integration over  $X(\mathbb{C})^{\{\text{internal vertices of } T\}}$ . Indeed,  $\kappa_T(W)$  is the exterior product of (65) and a smooth differential form on the product of copies of  $X(\mathbb{C})$  corresponding to the internal and  $S^*$ -decorated vertices, extended by  $\delta$ -function to (62).

2. We may assume that the  $S$ -decorated vertices are decorated by divisors supported on  $S$  by extending the Hodge correlator map by linearity.

3. For a given measure  $\mu$ , the corresponding Green function  $G_\mu$  is defined up to adding a constant. The Hodge correlators depend on it. However this dependence is minimal. If  $W$  has at least one degree zero divisor  $D$  as a factor, the corresponding Hodge correlator does not depend on the choice of the constant. Indeed, we may assume that in the form  $\omega_m$  we do not differentiate the Green function  $G_E$  assigned to the edge  $E$  decorated by  $D$ .

## 2.4 Shuffle relations for Hodge correlators

Our goal is the following shuffle relations for Hodge correlators:

**Proposition 2.7** *Let  $v_i \in V_{X,S^*}^\vee$ . Then for any  $p, q \geq 1$  one has*

$$\sum_{\sigma \in \Sigma_{p,q}} \text{Cor}_{\mathcal{H}} \mathcal{C}(v_0 \otimes v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(p+q)}) = 0, \quad (67)$$

where the sum is over all  $(p, q)$ -shuffles  $\sigma \in \Sigma_{p,q}$ .

To prove this we interpret the “sum over plane trivalent trees” construction using cyclic operades.

*Trees and the cyclic Lie co-operad.* Let  $\mathcal{C}Ass_\bullet$  be the cyclic associative operad. This means, in particular, the following. Let  $Ass_m\{v_1, \dots, v_m\}$  be the space of all associative words formed by the letters  $v_1, \dots, v_m$ , each used once. Then  $\mathcal{C}Ass_{m+1}$  is a vector space generated by the expressions

$$(A, v_0), \quad \text{where } A \in Ass_m\{v_1, \dots, v_m\}; \quad (xy, z) = (x, yz), \quad (x, y) = (y, x),$$

i.e.  $(*, *)$  is an invariant scalar product on an associative algebra. So as a vector space  $\mathcal{C}Ass_{m+1}$  is isomorphic to the space of  $m$ -ary operations  $Ass_m$  in the associative operad, or simply speaking to  $Ass_m\{v_1, \dots, v_m\}$ .

Similarly let  $\mathcal{C}Lie_\bullet$  be the cyclic Lie operad. Let  $Lie_m\{v_1, \dots, v_m\}$  be the space of all Lie words formed by the letters  $v_1, \dots, v_m$ , each used once. Then as a vector space  $\mathcal{C}Lie_{m+1}$  is isomorphic to the space of  $m$ -ary operations in the Lie operad. We think about it as of the space generated by expressions

$$(L, v_0), \quad \text{where } L \in Lie_m\{v_1, \dots, v_m\}; \quad ([x, y], z) = (x, [y, z]), \quad (x, y) = (y, x),$$

i.e.  $(*, *)$  is an invariant scalar product on a Lie algebra.

**Example.** The space  $\mathcal{C}Lie_3$  is one dimensional. It is generated by  $([v_1, v_2], v_0)$ . The space  $\mathcal{C}Ass_3$  is two dimensional, generated by  $(v_1 v_2, v_0)$  and  $(v_2 v_1, v_0)$ .

Let  $\mathcal{C}Ass_{m+1}^*$  (resp.  $\mathcal{C}Lie_{m+1}^*$ ) be the  $\mathbb{Q}$ -vector space dual to  $\mathcal{C}Ass_{m+1}$  (resp.  $\mathcal{C}Lie_{m+1}$ ). We define the subspace of *shuffle relations* in  $\mathcal{C}Ass_{m+1}^*$  as follows. Let  $(v_0 v_1 \dots v_m)^* \in \mathcal{C}Ass_{m+1}^*$  be a functional whose value on  $(v_1 \dots v_m, v_0)$  is 1 and on  $(v_{i_1} \dots v_{i_m}, v_0)$  is zero if  $\{i_1, \dots, i_m\} \neq \{1, 2, \dots, m\}$  as ordered sets. The subspace of shuffle relations is generated by the expressions

$$\sum_{\sigma \in \Sigma_{k, m-k}} (v_0 v_{\sigma(1)} \dots v_{\sigma(m)})^*; \quad 1 \leq k \leq m-1.$$

**Lemma 2.8** *There is a canonical isomorphism*

$$\mathcal{C}Lie_{m+1}^* = \frac{\mathcal{C}Ass_{m+1}^*}{\text{Shuffle relations}}. \quad (68)$$



**Proof.** Notice that  $\mathcal{L}ie_m\{v_1, \dots, v_m\} \subset \mathcal{A}ss_m\{v_1, \dots, v_m\}$  is the subspace of primitive elements for the coproduct  $\Delta$ ,  $\Delta(v_i) = v_i \otimes 1 + 1 \otimes v_i$ . Thus its dual  $\mathcal{L}ie_m^*\{v_1, \dots, v_m\}$  is the quotient of  $\mathcal{A}ss_m^*\{v_1, \dots, v_m\}$  by the shuffle relations.

*The tree complex ([K]).* Denote by  $T_{(m)}^i$  the abelian group generated by the isomorphism classes of pairs  $(T, \text{Or}_T)$  where  $T$  is a tree (not a plane tree!) with  $2m - i$  edges and with  $m + 1$  ends decorated by the set  $\{v_0, \dots, v_m\}$ . Here  $\text{Or}_T$  is an orientation of the tree  $T$ . The only relation is that changing the orientation of the tree  $T$  amounts to changing the sign of the generator.

There is a differential  $d : T_{(m)}^i \longrightarrow T_{(m)}^{i+1}$  defined by shrinking of internal edges of a tree  $T$ :

$$(T, \text{Or}_T) \longmapsto \sum_{\text{internal edges } E \text{ of } T} (T/E, \text{Or}_{T/E}).$$

Here if  $\text{Or}_T = E \wedge E_1 \wedge \dots$ , then  $\text{Or}_{T/E} := E_1 \wedge \dots$ .

The group  $T_{(m)}^i$  is a free abelian group with a basis; the basis vectors are defined up to a sign. Thus there is a perfect pairing  $T_{(m)}^i \otimes T_{(m)}^i \longrightarrow \mathbb{Z}$ , and we may identify the group  $T_{(m)}^i$  with its dual  $(T_{(m)}^i)^* := \text{Hom}(T_{(m)}^i, \mathbb{Z})$ . Consider the map  $w$

$$(T_{(m)}^1)^* \xrightarrow{w} \mathcal{C}\mathcal{L}ie_{m+1} \xrightarrow{i} \mathcal{C}\mathcal{A}ss_{m+1}.$$

defined as follows. Take an element  $(T, \text{Or}_T)$  represented by a 3-valent tree  $T$  decorated by  $v_0, v_1, \dots, v_m$  and an orientation  $\text{Or}_T$  of  $T$ . Make a decorated tree rooted at  $v_0$  out of the tree  $T$ . Define a Lie word in  $v_1, \dots, v_m$  using the oriented tree. We are getting an element  $w(T) \in \mathcal{C}\mathcal{L}ie_{m+1}$ . Viewing it as an associative word, we arrive at the element  $i \circ w(T)$ . There are the dual maps

$$\mathcal{C}\mathcal{A}ss_{m+1}^* \xrightarrow{i^*} \mathcal{C}\mathcal{L}ie_{m+1}^* \xrightarrow{w^*} (T_{(m)}^1)^*. \quad (69)$$

The following lemma offers an explanation of the “sum over plane trivalent trees” construction.

**Lemma 2.9** *The element  $w^* \circ i^*(v_0 v_1 \dots v_m)^*$  is the sum of all plane trivalent trees decorated by the cyclic word  $\mathcal{C}(v_0 v_1 \dots v_m)$ , and equipped with the canonical orientation induced by the clockwise orientation of the plane.*

For any  $p, q \geq 1$  with  $p + q = m$  one has

$$w^* \circ i^* \sum_{\sigma \in \Sigma_{p,q}} (v_0 v_{\sigma(1)} \dots v_{\sigma(m)})^* = 0,$$

where the sum is over all  $(p, q)$ -shuffles. Indeed, we apply  $w^*$  to the expression which is zero by Lemma 2.8. This plus Lemma 2.9 immediately imply

**Corollary 2.10** *The element of  $(T_{(m)}^1)^*$  given by the sum over all shuffles  $\sigma \in \Sigma_{p,q}$  of the sum of all plane trivalent trees decorated by  $(v_0 v_{\sigma(1)} \dots v_{\sigma(m)})$  is zero.*

**Proof of Proposition 2.7.** Corollary 2.10 implies that the integral in (67) is taken over zero cycle.

**Remark.** It was proved in [GK] that the following complex is exact:

$$\mathcal{C}Lie_{m+1}^* \xrightarrow{w^*} T_{(m)}^1 \xrightarrow{d} T_{(m)}^2 \xrightarrow{d} \dots \xrightarrow{d} T_{(m)}^{m-2}. \quad (70)$$

### 3 The twistor transform and Hodge DGA of a complex projective variety

#### 3.1 Hodge complexes

*The Dolbeaut complex of a variation of Hodge structures.* Let  $(\mathcal{L}, \nabla)$  be a variation of real Hodge structures over  $X(\mathbb{C})$ . Its real de Rham complex  $\mathcal{A}^*(\mathcal{L}) := \mathcal{A}^* \otimes_{\mathbb{R}} \mathcal{L}$  has a differential  $\mathbf{d}$ . Recall the Hodge decomposition

$$\mathcal{L}_{\mathbb{C}} := \mathcal{L} \otimes_{\mathbb{R}} \mathbb{C} = \oplus_{k,l} \mathcal{L}^{k,l}.$$

To define the Dolbeaut bicomplex of  $\mathcal{L}$ , set

$$\mathcal{A}^{*,*}(\mathcal{L}) = \oplus_{s,t} \mathcal{A}^{s,t}(\mathcal{L}), \quad \mathcal{A}^{s,t}(\mathcal{L}) := \oplus_{s=a+k, t=b+l} \mathcal{A}^{a,b} \otimes_{\mathbb{R}} \mathcal{L}^{k,l}.$$

This space is bigraded by the *Dolbeaut bidegree*  $(s, t)$ . It is equipped with the differential  $\mathbf{d}$ . Now we use the fact that  $\mathcal{L}$  is a variation of Hodge structures. The Griffiths transversality condition just means that the components the differential  $\mathbf{d}$  are of Dolbeaut bidegrees  $(1, 0)$  and  $(0, 1)$ . Let  $\mathbf{d} = \partial' + \partial''$  be the decomposition into the components of Dolbeaut bidegrees  $(1, 0)$  and  $(0, 1)$ . Decomposing  $\mathbf{d}^2 = 0$  into the components of Dolbeaut bidegrees  $(2, 0)$ ,  $(1, 1)$  and  $(0, 2)$  we get

$$\partial'^2 = \partial''^2 = \partial' \partial'' + \partial'' \partial' = 0.$$

So we get the Dolbeaut bicomplex  $(\mathcal{A}^{*,*}(\mathcal{L}), \partial', \partial'')$  of a variation of Hodge structures. Its total bicomplex is isomorphic to the complexification of the de Rham complex.

Let  $\mathbf{d} = \partial + \bar{\partial}$  be the decomposition into the holomorphic and antiholomorphic components. Thanks to the Griffiths transversality we can write

$$\begin{aligned} \partial &= \partial_0 + \lambda^{1,0}, & \lambda^{1,0} &\in \text{Hom}(\mathcal{L}^{*,*}, \mathcal{L}^{*-1,*+1}) \otimes \Omega^{1,0}, \\ \bar{\partial} &= \bar{\partial}_0 + \lambda^{0,1}, & \lambda^{0,1} &\in \text{Hom}(\mathcal{L}^{*,*}, \mathcal{L}^{*+1,*-1}) \otimes \Omega^{0,1}. \end{aligned}$$

Here  $\partial_0$  and  $\bar{\partial}_0$  are the components of  $\partial$  and  $\bar{\partial}$  preserving the Hodge decomposition. Then

$$\partial' = \partial_0 + \lambda^{0,1}, \quad \partial'' = \bar{\partial}_0 + \lambda^{1,0};$$

Set

$$\mathbf{d}^{\mathbb{C}} := \partial' - \partial'' = \partial_0 - \lambda^{1,0} - \bar{\partial}_0 + \lambda^{0,1}, \quad \mathbf{d} \mathbf{d}^{\mathbb{C}} = -2\partial' \partial''.$$

Consider the following subspace of the Dolbeaut bicomplex  $\mathcal{A}^{*,*}(\mathcal{L})$  sitting in the non-positive degrees, see Fig 5:

$$\tilde{\mathcal{C}}_{\mathcal{H}}(\mathcal{L}) := \bigoplus_{s,t \leq -1} \mathcal{A}^{s,t}(\mathcal{L}) \bigoplus \mathcal{A}_{\text{cl}}^{0,0}(\mathcal{L}). \quad (71)$$

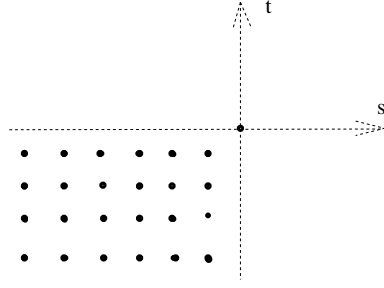


Figure 5: The Dolbeaut bigrading  $(s, t)$  on the Hodge complex.

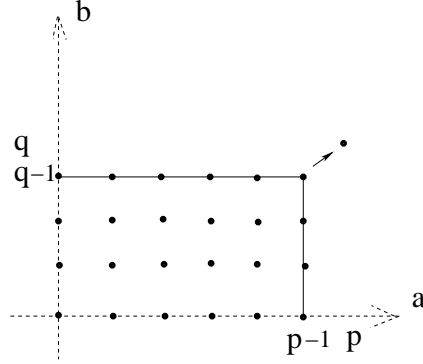


Figure 6: The de Rham bigrading  $(a, b)$  on the Hodge  $(-p, -q)$ -component of the Hodge complex.

Here  $\mathcal{A}_{\text{cl}}^{0,0}(\mathcal{L})$  is the subspace of  $\mathbf{d}$ -closed forms in  $\mathcal{A}^{0,0}(\mathcal{L})$ .

The de Rham bigrading  $(a, b)$  of its Hodge  $(-p, -q)$ -component is shown on Fig. 6.

Let us define a degree on  $\tilde{\mathcal{C}}_{\mathcal{H}}(\mathcal{L})$ . Given an  $(a, b)$ -form  $\varphi$  in the Hodge  $(-p, -q)$ -part of (71), set

$$\deg(\varphi) = \begin{cases} a + b & \text{if } (a, b) = (p, q), \\ a + b + 1 & \text{otherwise.} \end{cases} \quad (72)$$

We define a degree 1 differential  $D$  on  $\tilde{\mathcal{C}}_{\mathcal{H}}(\mathcal{L})$  by setting

$$D\varphi = \begin{cases} -\frac{1}{2}\mathbf{d}\mathbf{d}^c & \text{if } (s, t) = (-1, -1), \\ \partial'\varphi & \text{if } s < -1, t = -1, \\ \partial''\varphi & \text{if } s = -1, t < -1, \\ \mathbf{d}\varphi & \text{otherwise.} \end{cases} \quad (73)$$

So unless  $(s, t) = (-1, -1)$ , the map  $D$  is the de Rham differential  $\mathbf{d}$  truncated to fit the rectangle  $s, t \leq -1$ . We get a complex  $\tilde{\mathcal{C}}_{\mathcal{H}}^{\bullet}(\mathcal{L})$ .

Let  $N$  be a linear operator on  $\tilde{\mathcal{C}}_{\mathcal{H}}^{\bullet}(\mathcal{L})$ , acting on a homogeneous element  $\varphi$  of the Hodge bidegree  $(-p, -q)$  as

$$N := \text{multiplication by } N_{\varphi} := p + q - \deg(\varphi). \quad \text{So } N_{\varphi} \geq 0.$$

**Convention.** Below  $\varphi_{s,t}$  always denotes a homogeneous element of the Dolbeaut bidegree  $(-s, -t)$  in the Hodge complex, so  $s, t \geq 0$ .

The operator  $N$  depends only on the Dolbeaut bigrading: for a homogeneous element  $\varphi_{s,t}$  we have

$$N\varphi_{s,t} = \begin{cases} (s+t-1)\varphi_{s,t} & \text{if } (s,t) > (0,0). \\ 0 & \text{otherwise} \end{cases}$$

Let us define a subcomplex

$$\mathcal{C}_{\mathcal{H}}^{\bullet}(\mathcal{L}) \subset \tilde{\mathcal{C}}_{\mathcal{H}}^{\bullet}(\mathcal{L}),$$

which we call the *Hodge complex* of  $\mathcal{L}$ . A homogeneous element  $\varphi \in \tilde{\mathcal{C}}_{\mathcal{H}}^{\bullet}(\mathcal{L})$  is in  $\mathcal{C}_{\mathcal{H}}^{\bullet}(\mathcal{L})$  if and only if

$$\bar{\varphi} = \begin{cases} -\varphi & \text{if } N_{\varphi} > 0, \\ \varphi & \text{if } N_{\varphi} = 0. \end{cases} \quad (74)$$

Here the bar is the complex conjugation provided by the ones on the Dolbeaut bicomplex and  $\mathcal{L}_{\mathbb{C}}$ .

**Proposition 3.1** *The complex  $\mathcal{C}_{\mathcal{H}}^{\bullet}(\mathcal{L})$  is quasiisomorphic to  $\mathrm{RHom}_{\mathrm{Sh}_{\mathrm{Hod}(X)}}(\mathbb{R}(0), \mathcal{L})$ .*

**Proof.** One can calculate  $\mathrm{RHom}_{\mathrm{Sh}_{\mathrm{Hod}(X)}}(\mathbb{R}(0), \mathcal{L})$  in two steps. First, take the real de Rham complex  $\mathcal{A}^*(\mathcal{L})$  of  $\mathcal{L}$ . It has a (split) mixed  $\mathbb{R}$ -Hodge structure. Indeed, its complexification coincides with the Dolbeaut bicomplex of  $\mathcal{L}_{\mathbb{C}}$ . The latter is bigraded, and so gives rise to a mixed  $\mathbb{R}$ -Hodge structure in the trivial standard way. Notice that the weight and Hodge filtration on the *complex*  $\mathcal{A}^*(\mathcal{L})$  are obtained from the ones on the *vector space*  $\mathcal{A}^*(\mathcal{L})$  by the standard procedure, so that the differential is strictly compatible with the weight filtration.

Then calculate  $\mathrm{RHom}_{\mathrm{MHS}/\mathbb{R}}(\mathbb{R}(0), \mathcal{A}^*(\mathcal{L}))$  in the category of mixed  $\mathbb{R}$ -Hodge structures using the following standard result. Given a complex  $H$  of mixed  $\mathbb{R}$ -Hodge structures

$$\mathrm{RHom}_{\mathrm{MHS}/\mathbb{R}}(\mathbb{R}(0), H) = \mathrm{Cone}\left(W_0 H \oplus F^0(W_0 H)_{\mathbb{C}} \longrightarrow (W_0 H)_{\mathbb{C}}\right). \quad (75)$$

**Lemma 3.2** *Complex (75) for the complex of mixed  $\mathbb{R}$ -Hodge structure  $H = \mathcal{A}^*(\mathcal{L})$  is quasiisomorphic to  $\mathcal{C}_{\mathcal{H}}^{\bullet}(\mathcal{L})$ .*

**Proof.** Let  $f^* : X^* \rightarrow Y^*$  be a morphism of complexes such that  $f^i$  is injective for  $i \leq k$  and surjective for  $i \geq k$ . Then there is a complex

$$Z^* := \mathrm{Coker} f^{<k}[-1] \longrightarrow \mathrm{Ker} Y^{>k}$$

with a differential  $D : \mathrm{Coker} f^{k-1} \rightarrow \mathrm{Ker} Y^{k+1}$  defined by a diagram chase ([G8], Proposition 2.1). Thanks to Lemma 2.2 in loc. cit., the complex  $Z^*$  is canonically isomorphic to  $\mathrm{Cone}(X^* \rightarrow Y^*)$ .

Let us apply this construction in our case. One has

$$W_0 H^* = W_0 \mathcal{A}^*(\mathcal{L}) = \bigoplus_{s+t \leq 0} \mathcal{A}^{s,t} \mathcal{L},$$

$$F^0(W_0 H)_{\mathbb{C}} = W_0 \mathcal{A}^{*,*}(\mathcal{L}) = \bigoplus_{s \geq 0} \mathcal{A}^{s,t} \mathcal{L}.$$

The cokernel of the map  $W_0(\mathcal{A}^*(\mathcal{L})) \rightarrow W_0(\mathcal{A}^*(\mathcal{L}))_{\mathbb{C}}$  is the imaginary part. The lemma, and hence the proposition follow from this.

### 3.2 The twistor transform and the Hodge DGA

Let  $\mathcal{V}$  be a variation of real Hodge structures on  $X(\mathbb{C})$ . Let us define a linear map of vector spaces

$$\mathbf{1}^* : \mathcal{C}_{\mathcal{H}}^{\bullet}(\mathcal{V}) \longrightarrow \mathcal{A}^{\bullet}(\mathcal{V}), \quad \mathbf{1}^* \varphi := \begin{cases} \mathbf{d}^{\mathbb{C}} \varphi & \text{if } N_{\varphi} > 0, \\ \varphi & \text{if } N_{\varphi} = 0. \end{cases}$$

The map  $\mathbf{1}^*$  preserves the degree. Indeed, we apply  $\mathbf{d}^{\mathbb{C}}$  to those homogeneous elements whose degree in the Hodge complex is one plus their de Rham degree, and do not change the homogeneous elements whose degree in the Hodge complex coincides with their de Rham degree. Since the local system  $\mathcal{V}$  is real, its de Rham complex inherits a real structure. The map  $\mathbf{1}^*$  is a real map:

$$\overline{\mathbf{1}^* \varphi} = \mathbf{1}^* \varphi. \quad (76)$$

Recall the canonical parameter  $u$  on the twistor line  $\mathbb{R}$ .

**Definition 3.3** Set  $\psi := \mathbf{1}^* \varphi$ . The twistor transform is a linear map

$$\begin{aligned} \gamma : \mathcal{C}_{\mathcal{H}}^{\bullet}(\mathcal{V}) &\longrightarrow \mathcal{A}^{\bullet}(X(\mathbb{C}) \times \mathbb{R}, p^* \mathcal{V}), \\ \gamma : \varphi = \sum_{s,t} \varphi_{s,t} &\longmapsto \sum_{s,t \geq 0} (1-u)^s (1+u)^t \left( du \wedge (s+t+1) \varphi_{s+1,t+1} + \psi_{s,t} \right). \end{aligned} \quad (77)$$

One can write it in the homogeneous form, using the twistor plane  $\mathbb{C}^2$  with the coordinates  $(z, w)$  and the projection  $\pi : X(\mathbb{C}) \times \mathbb{C}^2 \rightarrow X(\mathbb{C})$  (Section 1.5). We define a *homogeneous twistor transform*

$$\begin{aligned} \hat{\gamma} : \tilde{\mathcal{C}}_{\mathcal{H}}^{\bullet}(\mathcal{V}) &\longrightarrow \mathcal{A}^{\bullet}(X(\mathbb{C}) \times \mathbb{C}^2, \pi^* \mathcal{V}), \\ \hat{\gamma} : \varphi = \sum_{s,t} \varphi_{s,t} &\longmapsto \sum_{s,t \geq 0} z^s w^t \left( (zdw - wdz)/2 \wedge (s+t+1) \varphi_{s+1,t+1} + \psi_{s,t} \right). \end{aligned} \quad (78)$$

Recall the involution  $\sigma : (z, w) \mapsto (\bar{w}, \bar{z})$  of the twistor plane, and the complex conjugation  $c$ . Then

$$(c \circ \sigma)^* \hat{\gamma}(\varphi) = \hat{\gamma}(\varphi) \text{ if and only if } \varphi \in \mathcal{C}_{\mathcal{H}}^{\bullet}(\mathcal{V}).$$

Restricting (78) to the line  $z + w = 2$  we get (77). The twistor transform is clearly functorial in  $\mathcal{V}$ .

**First steps of the proof of Theorem 1.6.** The injectivity of  $\gamma$  is clear. We have to check the following three statements:

- (i) The map  $\gamma$  preserves the degree.
- (ii) There is a product  $*$  on Hodge complexes such that for any variations  $\mathcal{V}_1$  and  $\mathcal{V}_2$  the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{H}}^{\bullet}(\mathcal{V}_1) \otimes \mathcal{C}_{\mathcal{H}}^{\bullet}(\mathcal{V}_2) & \xrightarrow{*} & \mathcal{C}_{\mathcal{H}}^{\bullet}(\mathcal{V}_1 \otimes \mathcal{V}_2) \\ \downarrow \gamma & & \gamma \downarrow \\ \mathcal{A}^{\bullet}(p^* \mathcal{V}_1) \otimes \mathcal{A}^{\bullet}(p^* \mathcal{V}_2) & \xrightarrow{\wedge} & \mathcal{A}^{\bullet}(p^* \mathcal{V}_1 \otimes p^* \mathcal{V}_2) \end{array} \quad (79)$$

(iii) There is a differential  $\delta$  on the Hodge complex  $\mathcal{C}_{\mathcal{H}}^{\bullet}(\mathcal{V})$  such that the map  $\gamma$  transforms  $\delta$  into the de Rham differential  $\mathbf{d} + d_u$  on the de Rham complex of  $p^*\mathcal{V}$  on  $X(\mathbb{C}) \times \mathbb{R}$ , i.e. we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{H}}^{\bullet}(\mathcal{V}) & \xrightarrow{\delta} & \mathcal{C}_{\mathcal{H}}^{\bullet}(\mathcal{V}) \\ \downarrow \gamma & & \gamma \downarrow \\ \mathcal{A}^{\bullet}(p^*\mathcal{V}) & \xrightarrow{\mathbf{d}+d_u} & \mathcal{A}^{\bullet}(p^*\mathcal{V}) \end{array}$$

The property (i) is clear. Indeed, since the map  $\mathbf{1}^*$  preserves the degrees, the second term in (77) has the same property. Let us look at the first term in (77). There one has  $N_{\varphi_{s+1,t+1}} > 0$ . Therefore the wedge product with  $du$  compensates the fact that the degree of  $\varphi_{s,t}$  in the Hodge complex is bigger than its de Rham degree by 1.

Below we introduce a product  $*$  and a differential  $\delta$  which match the product and the differential on the de Rham DG algebra under the twistor transform. Then the claim that  $(\mathcal{D}_X, *, \delta)$  is a commutative DG algebra is straightforward since the twistor transform is injective.

The formulas for the product  $*$  and the differential  $\delta$  are not complicated. It is easy to check directly that  $\delta^2 = 0$  and that the product is graded commutative. However without using the twistor transform a proof of the Leibniz rule for  $\delta$  requires a messy calculation. This is hardly surprising: hiding the role of the twistor line and hence the differential  $d_u$  obscures the story.

**A product on  $\mathcal{D}_X$ .** We use a shorthand  $\overline{\varphi}$  for the degree of  $\varphi$  in the Hodge complex.

**Definition 3.4**

$$\varphi_1 * \dots * \varphi_m :=$$

$$\begin{cases} \varphi_1 \wedge \dots \wedge \varphi_m & \text{if } \sum N_{\varphi_i} = 0, \\ N^{-1} \left( \sum_{i=1}^k (-1)^{\overline{\varphi}_i(\overline{\varphi}_1 + \dots + \overline{\varphi}_{i-1})} N \varphi_i \wedge \mathbf{1} * \varphi_1 \wedge \dots \wedge \widehat{\mathbf{1} * \varphi_i} \wedge \dots \wedge \mathbf{1} * \varphi_m \right) & \text{otherwise.} \end{cases}$$

**Proposition 3.5** (i) *One has*

$$\mathbf{1} * (\varphi_1 * \dots * \varphi_m) = \mathbf{1} * \varphi_1 \wedge \dots \wedge \mathbf{1} * \varphi_m. \quad (80)$$

(ii) *The product  $*$  makes the diagram (79) commutative.*

(iii) *The pair  $(\mathcal{D}_X, *)$  is a supercommutative algebra in the category  $\text{Hod}_X$ .*

**Proof.** (i) Obvious.

(ii) The commutativity of the diagram in the case when  $\sum N_{\varphi_i} = 0$  is obvious. So below we may assume that  $\sum N_{\varphi_i} > 0$ . Take the  $\wedge$ -product of several expressions (77). It contains at most one  $du$ . Look at its  $du$ -component. Observe that the degree of  $\mathbf{1} * \varphi$  in the Hodge complex coincides with its degree in the de Rham complex, as well as with the de Rham degree of  $du \wedge N \varphi_i$ , provided  $N \varphi_i \neq 0$ . So moving  $du \wedge N \varphi_i$  to the left we get

$$du \wedge \sum_{i=1}^k (-1)^{\overline{\varphi}_i(\overline{\varphi}_1 + \dots + \overline{\varphi}_{i-1})} N \varphi_i \wedge \mathbf{1} * \varphi_1 \wedge \dots \wedge \widehat{\mathbf{1} * \varphi_i} \wedge \dots \wedge \mathbf{1} * \varphi_m.$$

This is nothing else but  $du \wedge N(\varphi_1 * \dots * \varphi_m)$ . So we recovered Definition 3.4 for the  $*$ -product in the case  $\sum N_{\varphi_i} > 0$ . The claim that the second term in (77) is multiplicative is equivalent to (i).

(iii) Follows from the injectivity of the twistor transform. The proposition is proved.

*Explicit formulas.*

$$\varphi_1 * \varphi_2 = \begin{cases} \varphi_1 \wedge \varphi_2 & \text{if } N_{\varphi_1} = N_{\varphi_2} = 0, \\ N^{-1} \left( N\varphi_1 \wedge \varphi_2 + (-1)^{\bar{\varphi}_1 \bar{\varphi}_2} N\varphi_2 \wedge \varphi_1 \right) & \text{if } N_{\varphi_1} = 0 \text{ or } N_{\varphi_2} = 0, \\ & \text{but } N_{\varphi_1} + N_{\varphi_2} > 0, \\ N^{-1} \left( N\varphi_1 \wedge \mathbf{d}^c \varphi_2 + (-1)^{\bar{\varphi}_1 \bar{\varphi}_2} N\varphi_2 \wedge \mathbf{d}^c \varphi_1 \right) & \text{if } N_{\varphi_1} > 0, N_{\varphi_2} > 0. \end{cases} \quad (81)$$

**A differential on  $\mathcal{D}_X$ .** Let  $\mu$  be a linear operator on  $\mathcal{C}_{\mathcal{H}}^{\bullet}(\mathcal{L})$  acting on a homogeneous element  $\varphi_{s,t}$  as

$$\mu(\varphi_{s,t}) := \begin{cases} \binom{s+t-2}{s-1} \varphi_{s,t} & \text{if } s+t \geq 2, \\ \varphi_{s,t} & \text{otherwise.} \end{cases} \quad (82)$$

**Definition 3.6**  $\delta := -\mu \circ 2D \circ \mu^{-1}$ .

Let  $N'$  and  $N''$  be the operators acting on an element of the Dolbeaut bidegree  $(-s, -t)$  of the Hodge complex  $\mathcal{C}_{\mathcal{H}}^{\bullet}(\mathcal{L})$  as follows:

$$N' := \text{multiplication by } s-1, \quad N'' := \text{multiplication by } t-1.$$

Observe that  $N = N' + N'' + 1 = s+t-1$  provided  $s, t \leq -1$ .

**Lemma 3.7** *One has*

$$\delta(\varphi) = \begin{cases} 0 & \text{if } N_{\varphi} = 0 \\ -2\partial'\partial''\varphi & \text{if } N_{\varphi} = 1 \\ -2N^{-1}(\partial'N' + \partial''N'')\varphi & \text{if } N_{\varphi} > 1. \end{cases} \quad (83)$$

**Proof.** In the case  $s+t > 2$  for a homogeneous element  $\varphi_{s,t}$  we have

$$\delta(\varphi_{s,t}) = \frac{-2}{s+t-2} \left( (s-1)\partial' + (t-1)\partial'' \right) \varphi_{s,t}. \quad (84)$$

Write  $\delta = \delta' + \delta''$  as a sum of the components of bidegrees  $(1,0)$  and  $(0,1)$ . We prove the claim for  $\delta'$ , and use the complex conjugation to get the rest. Then we have

$$-\mu^{-1}\delta'(\varphi_{s,t}) = \binom{s+t-3}{s-2}^{-1} \frac{s-1}{s+t-2} \partial' \varphi_{s,t} = \binom{s+t-2}{s-1}^{-1} \partial' \varphi_{s,t} = \partial' \mu^{-1}(\varphi_{s,t}).$$

The case  $r+s \leq 2$  is obvious. The lemma is proved.

**Proof of Theorem 1.6: the key step.** Let us prove the remaining claim (iii). This is the only step which requires an elaborate computation. One has

$$\mathbf{d}\gamma(\varphi) = - \sum_{s,t \geq 0} (1-u)^s (1+u)^t du \wedge (s+t+1) \mathbf{d}\varphi_{s+1,t+1} \quad (85)$$

$$+ \sum_{s,t \geq 0} \frac{d}{du} [(1-u)^s(1+u)^t] du \wedge \psi_{s,t} \quad (86)$$

$$+ \sum_{s,t \geq 0} (1-u)^s(1+u)^t \mathbf{d}\psi_{s,t}. \quad (87)$$

Let us look first on the part of  $\mathbf{d}\gamma(\varphi)$  containing  $du$ , i.e. on the first two lines. In this case  $\psi = \mathbf{d}^c\varphi$ . We examine its  $\partial'$ -component – the other one follows then by complex conjugation. We have

$$- \sum_{s,t \geq 1} (1-u)^{s-1}(1+u)^{t-1} du \wedge \left( (s+t-1)\partial'(\varphi_{s,t}) + [s(1+u) - t(1-u)](\partial'\varphi)_{s,t} \right) \quad (88)$$

$$+ \sum_{t > 0} \frac{d}{du} (1+u)^t du \wedge \psi_{0,t}. \quad (89)$$

Here (85) plus (86) with  $s, t > 0$  provides (88), and (86) with  $s = 0, t > 0$  provides (89). The term (86) with  $t = 0$  is killed by  $d/du$ . We have  $(\partial'\varphi)_{s,t} = \partial'(\varphi_{s+1,t})$ . Using this, we get

$$- \sum_{s \geq 2; t \geq 1} (1-u)^{s-2}(1+u)^{t-1} du \wedge \partial'(\varphi_{s,t}) \times \quad (90)$$

$$\left( (s+t-1)(1-u) + [(s-1)(1+u) - t(1-u)] \right) \quad (91)$$

$$- \sum_{t > 0} t(1+u)^{t-1} \partial'(\varphi_{1,t}) \wedge du + (89). \quad (92)$$

The left term of (92) corresponds to the case  $s = 1$  in (88). We claim that (92) = 0. Indeed, since  $t > 0$ ,  $\psi_{0,t} = (\partial'\varphi)_{0,t} = \partial'(\varphi_{1,t})$ . So the two terms in (92) cancel. Further, (91) =  $2(s-1)$ . Therefore we get

$$(90) \& (91) = - \sum_{s \geq 2; t \geq 1} (1-u)^{s-2}(1+u)^{t-1} du \wedge 2(s-1)\partial'(\varphi_{s,t}). \quad (93)$$

On the other hand, we have

$$\gamma\delta(\varphi) = \sum_{s,t \geq 0} (1-u)^s(1+u)^t \left( (s+t+1)du \wedge (\delta\varphi)_{s+1,t+1} + (\mathbf{1} * \delta\varphi)_{s,t} \right). \quad (94)$$

Using (84), we conclude that (93) coincides with the  $du$ -part in (94).

Let us compare now the  $du$ -free part of  $\mathbf{d}\gamma(\varphi)$ , that is (87), with the  $du$ -free part of  $\gamma\delta(\varphi)$ . The claim that they coincide is equivalent to the identity

$$\mathbf{d}(\mathbf{1} * \varphi) = \mathbf{1} * \delta\varphi. \quad (95)$$

If  $N_\varphi = 0$ , both parts are zero since in this case  $\mathbf{d}\varphi = 0$ . If  $N_\varphi = 1$ , both parts are  $\mathbf{d}\mathbf{d}^c\varphi$ . If  $N_\varphi > 1$ , the left hand side is  $\mathbf{d}\mathbf{d}^c\varphi = -2\partial\bar{\partial}\varphi$ . The right hand side of (95) is

$$\mathbf{1} * \delta\varphi = (\partial' - \partial'')\delta(\varphi) = \frac{-2(\partial' - \partial'')}{s+t-2} \left( (s-1)\partial' + (t-1)\partial'' \right) \varphi = -2\partial'\partial''\varphi.$$

Theorem 1.6 is proved.



**Concluding remarks.** (i) The constant function 1 on  $X(\mathbb{C})$  provides a canonical element

$$\mathbf{1}_{\mathbb{R}(1)} := 1 \otimes (2\pi i) \in \mathcal{C}_{\mathcal{H}}^1(\mathbb{R}(1)). \quad (96)$$

Given a variation of real Hodge structures  $\mathcal{L}$ , there is a map provided by the  $*$ -product with  $\mathbf{1}_{\mathbb{R}(1)}$ :

$$\tau : \mathcal{C}_{\mathcal{H}}^{\bullet}(\mathcal{L}) \longrightarrow \mathcal{C}_{\mathcal{H}}^{\bullet+1}(\mathcal{L}(1)), \quad \tau\varphi := \begin{cases} d^{\mathbb{C}}\varphi \otimes 2\pi i & \text{if } \varphi \text{ homogeneous, } N_{\varphi} > 0, \\ \varphi \otimes 2\pi i & \text{if } \varphi \text{ homogeneous, } N_{\varphi} = 0. \end{cases} \quad (97)$$

**Remark.** The linear map  $\mathbf{1}*$  is nothing else but the  $*$ -product with  $\mathbf{1}_{\mathbb{R}(1)}$  followed by multiplication by  $(2\pi i)^{-1}$  and the tautological linear embedding  $\mathcal{C}_{\mathcal{H}}^{\bullet+1}(\mathcal{L}) \hookrightarrow \mathcal{A}^{\bullet}(\mathcal{L})$ .

**Lemma 3.8** *One has  $\tau^2 = 0$ . The map  $\tau$  commutes with the differential  $\delta$ . So the Hodge cohomology groups form a complex:*

$$\dots \xrightarrow{\tau} \mathbf{H}_{\mathcal{H}}^{\bullet}(\mathcal{L}) \xrightarrow{\tau} \mathbf{H}_{\mathcal{H}}^{\bullet+1}(\mathcal{L}(1)) \xrightarrow{\tau} \mathbf{H}_{\mathcal{H}}^{\bullet+2}(\mathcal{L}(2)) \xrightarrow{\tau} \dots$$

**Proof.** Clearly  $\mathbf{1}_{\mathbb{R}(1)} * \mathbf{1}_{\mathbb{R}(1)} = d^{\mathbb{C}}1 \otimes (2\pi i)^2 = 0$ . This plus associativity of the  $*$ -product implies  $\tau^2 = 0$ . Since  $\delta\mathbf{1}_{\mathbb{R}(1)} = 0$ , the second statement follows from the Leibniz rule. The lemma is proved.

**Remark.** A similar, yet different, complex of Weil-étale cohomologies is used by S. Lichtenbaum.

(ii) *Algebra structures on  $\mathcal{A}_M^*[-1]$ .* Here is a family of graded commutative algebra structures on the shifted by one smooth de Rham complex  $\mathcal{A}_M^*[-1]$  of a complex manifold  $M$ . Let  $N$  be a linear operator on  $\mathcal{A}_M^*$  acting by a scalar on the subspace of forms of given degree: for a homogeneous form  $\alpha$  we have  $N(\alpha) = N_{\alpha} \cdot \alpha$ . Let us assume that for any two homogeneous forms  $\alpha_1, \alpha_2$  we have

$$N_{\alpha_1 \circ \alpha_2} = N_{\alpha_1} + N_{\alpha_2}. \quad (98)$$

Let  $\bar{\alpha}$  be the degree of  $\alpha$  in  $\mathcal{A}_M^*[-1]$ . We define a product  $\circ$  on  $\mathcal{A}_M^*[-1]$  by

$$N(\alpha_1 \circ \dots \circ \alpha_m) := \sum_{i=1}^k (-1)^{\bar{\alpha}_i(\bar{\alpha}_1 + \dots + \bar{\alpha}_{i-1})} N(\alpha_i) \wedge d^{\mathbb{C}}\alpha_1 \wedge \dots \wedge \widehat{N(\alpha_i)} \wedge \dots \wedge d^{\mathbb{C}}\alpha_m. \quad (99)$$

**Lemma 3.9** (i) *One has*

$$d^{\mathbb{C}}(\alpha_1 \circ \dots \circ \alpha_m) = d^{\mathbb{C}}\alpha_1 \wedge \dots \wedge d^{\mathbb{C}}\alpha_m. \quad (100)$$

(iii) *The product  $\circ$  provides  $\mathcal{A}_M^*[-1]$  with a structure of a graded commutative, associative algebra.*

**Proof.** (i) Obvious thanks to (98).

(ii) The product is supercommutative by the very definition. One has, using (i),

$$N((\alpha_1 \circ \alpha_2) \circ \alpha_3) = N(\alpha_1 \circ \alpha_2) \wedge d^{\mathbb{C}}\alpha_3 + (-1)^{\bar{\alpha}_3(\bar{\alpha}_1 + \bar{\alpha}_2)} N\alpha_3 \wedge d^{\mathbb{C}}(\alpha_1 \circ \alpha_2) = N(\alpha_1 \circ \alpha_2 \circ \alpha_3).$$

Similarly  $N(\alpha_1 \circ \alpha_2 \circ \alpha_3) = N(\alpha_1 \circ (\alpha_2 \circ \alpha_3))$ . The lemma is proved.

## 4 Twistor connections and variations of mixed $\mathbb{R}$ -Hodge structures

Below we assume that  $X$  is a smooth complex projective variety. All results of this section admit a rather straightforward generalization when  $X$  is a smooth complex variety, not necessarily projective. We have to use embedding of  $X$  in a regular compactification with a normal crossing divisor at infinity, and use the standard technique to extend our results.

Given a local system  $\mathcal{L}$  on a smooth manifold  $M$ , let  $\mathcal{L}_\infty := \mathcal{L} \otimes_{\mathbb{R}} \mathcal{A}_M$  be the corresponding smooth sheaf. If  $M$  is a complex manifold, denote by  $\mathcal{L}_{\mathcal{O}} := \mathcal{L} \otimes_{\mathbb{R}} \mathcal{O}_M$  the corresponding holomorphic sheaf.

**Definition 4.1** *A variation of mixed  $\mathbb{R}$ -Hodge structures over a complex variety  $X(\mathbb{C})$  is given by a real local system  $\mathcal{L}$  on  $X(\mathbb{C})$  equipped with the following data:*

- An increasing weight filtration  $W_\bullet \mathcal{L}$  of the local system  $\mathcal{L}$ ,
  - A decreasing Hodge filtration  $F^\bullet$  on the holomorphic vector bundle  $\mathcal{L}_{\mathcal{O}}$ ,
- satisfying the following conditions:
- The two filtrations induce at each fiber a mixed  $\mathbb{R}$ -Hodge structure,
  - The Griffiths transversality condition:  $\nabla^{1,0}(F^p) \subset F^{p-1} \otimes \Omega_X^1$ , where  $\nabla$  is the flat connection provided by the local system structure of  $\mathcal{L}$ .

The condition that  $F^p \mathcal{L}_{\mathcal{O}}$  is a holomorphic subbundle plus the Griffiths transversality just means that

$$\nabla(F^p) \subset F^{p-1} \otimes \Omega_X^1 \oplus F^p \otimes \overline{\Omega}_X^1.$$

The category  $\text{Hod}_X$  of variations of real Hodge structures on  $X(\mathbb{C})$  is a semi-simple abelian tensor category. Variations of mixed  $\mathbb{R}$ -Hodge structures on  $X(\mathbb{C})$  form an abelian tensor category  $\text{MHod}_X$ . It is a mixed category, equipped with a canonical tensor functor

$$\text{gr}_\bullet^W : \text{MHod}_X \longrightarrow \text{Hod}_X; \quad \mathcal{L} \longmapsto \text{gr}_\bullet^W \mathcal{L}.$$

Given a variation  $\mathcal{V}$  of real Hodge structures on  $X(\mathbb{C})$ , there is a canonical decomposition

$$\mathcal{V}_{\mathbb{C}} := \mathcal{V} \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p,q} \mathcal{V}^{p,q}. \tag{101}$$

Here  $\bigoplus_{p+q=n} \mathcal{V}^{p,q}$  is a pure complex Hodge structure of weight  $n$ .

Our goal is to equip the object (101) with an additional datum, the *Green datum*, which allows to recover the original variation of mixed  $\mathbb{R}$ -Hodge structures in a functorial way.

## 4.1 Green data and twistor connections

Let  $\mathcal{V}$  be a variation of real Hodge structures on  $X(\mathbb{C})$ . Let us introduce a DG Lie algebra of derived Hodge endomorphisms of  $\mathcal{V}$ . Let  $\text{End}\mathcal{V}$  be the local system of endomorphisms of the local system  $\mathcal{V}$ . Denote by  $\mathbf{d}$  the flat connection on the sheaf  $\text{End}\mathcal{V}_\infty$ . The Hodge complex  $\mathcal{C}_\mathcal{H}^\bullet(\text{End}\mathcal{V})$  has a DG Lie algebra structure: the commutator is given by the composition

$$\mathcal{C}_\mathcal{H}^\bullet(\text{End}\mathcal{V}) \otimes \mathcal{C}_\mathcal{H}^\bullet(\text{End}\mathcal{V}) \xrightarrow{*} \mathcal{C}_\mathcal{H}^\bullet(\text{End}\mathcal{V} \otimes \text{End}\mathcal{V}) \xrightarrow{[\cdot, \cdot]} \mathcal{C}_\mathcal{H}^\bullet(\text{End}\mathcal{V})$$

where the first map is the  $*$ -product, and the second is induced by the commutator map on  $\text{End}\mathcal{V}$ .

**Definition 4.2** *A Green datum on a variation of real Hodge structures  $\mathcal{V}$  is a degree 1 element*

$$\mathcal{G} \in \mathcal{C}_\mathcal{H}^1(\text{End}\mathcal{V}) \tag{102}$$

*of the DG Lie algebra  $\mathcal{C}_\mathcal{H}^\bullet(\text{End}\mathcal{V})$  which satisfies a Maurer-Cartan equation*

$$\delta\mathcal{G} + [\mathcal{G}, \mathcal{G}] = 0. \tag{103}$$

*Twistor connections.* Recall the projection  $p : X(\mathbb{C}) \times \mathbb{R} \rightarrow X(\mathbb{C})$  and the parameter  $u$  on the twistor line  $\mathbb{R}$ . A datum  $\mathcal{G}$  as in (102) provides a *twistor connection*  $\nabla_\mathcal{G}$  on  $p^*\mathcal{V}_\infty$ :

$$\nabla_\mathcal{G} := \mathbf{d} + \gamma(\mathcal{G}) \tag{104}$$

The crucial fact about it is the following:

**Proposition 4.3** *A datum  $\mathcal{G}$ , see (102), on a variation of real Hodge structures  $\mathcal{V}$  satisfies the Maurer-Cartan equation (103) if and only if the twistor connection  $\nabla_\mathcal{G}$  on  $X(\mathbb{C}) \times \mathbb{R}$  is flat.*

**Proof.** Follows immediately from Theorem 1.6.

Tensor product of two flat twistor connections is again a flat twistor connection. Therefore flat twistor connections, or, equivalently, Green data, form an abelian tensor category  $\tilde{\mathcal{G}}_\mathcal{H}(X)$ .

Explicitly, the objects of the category  $\tilde{\mathcal{G}}_\mathcal{H}(X)$  are pairs  $(\mathcal{V}, \mathcal{G})$ , where  $\mathcal{V}$  is a variation of real Hodge structures on  $X(\mathbb{C})$  and  $\mathcal{G}$  is a Green datum on it. The morphisms are morphisms of variations of Hodge structures  $\mathcal{V} \rightarrow \mathcal{V}'$  commuting with the Green data. The tensor product is given by  $(\mathcal{V}, \mathcal{G}) \otimes (\mathcal{V}', \mathcal{G}') = (\mathcal{V} \otimes \mathcal{V}', \mathcal{G} \otimes \text{Id}_{\mathcal{V}'} + \text{Id}_\mathcal{V} \otimes \mathcal{G}')$ .

**Lemma 4.4** *The category  $\tilde{\mathcal{G}}_\mathcal{H}(X)$  is canonically equivalent to the category  $\mathcal{G}_\mathcal{H}(X)$  of comodules over the Lie coalgebra  $\mathcal{L}_{\mathcal{H}, X}$  in the category  $\text{Hod}_X$ .*

We prove this lemma in Section 4.3 below. We will not use it before.

**Theorem 4.5** *There is a canonical equivalence of tensor categories between the category  $\text{MHod}_X$  of variations of mixed  $\mathbb{R}$ -Hodge structures on  $X(\mathbb{C})$  and the category  $\tilde{\mathcal{G}}_\mathcal{H}(X)$ .*

**Explicit formulas.** Before we start a proof of Theorem 4.5, let us elaborate these definitions. The bigrading on  $\mathcal{V}_{\mathbb{C}}$  induces a bigrading on the local system  $\text{End}\mathcal{V}_{\mathbb{C}}$ : its bidegree  $(-p, -q)$  component  $\text{End}^{-p, -q}\mathcal{V}$  consists of linear maps  $A$  such that  $A\mathcal{V}^{a, b} \subset \mathcal{V}^{a-p, b-q}$ .

**Lemma 4.6** *An element  $\mathcal{G}$  in (102) is given by the following datum:*

- A collection of linear operators

$$G_{p, q} \in \text{End}^{-p, -q}\mathcal{V}_{\infty}, \quad p, q \geq 1, \quad \overline{G}_{p, q} = -G_{p, q}, \quad (105)$$

- A real (i.e. invariant under the complex conjugation) closed 1-form with values in  $\text{End}\mathcal{V}$

$$\nu = \nu^{1, 0} + \nu^{0, 1} \in \Omega^1 \otimes \text{End}^{-1, 0}\mathcal{V}_{\infty} \oplus \overline{\Omega}^1 \otimes \text{End}^{0, -1}\mathcal{V}_{\infty}, \quad \overline{\nu} = \nu, \quad \mathbf{d}\nu = 0. \quad (106)$$

We call the operator  $G = \sum G_{p, q}$  the *Green operator*.

**Proof.** An element of  $\mathcal{C}_{\mathcal{H}}^1(\text{End}\mathcal{V})$  is given by a pair  $\{G, \nu\}$ , where  $G = \sum G_{p, q}$  is a function,  $\nu = \sum \nu_{p', q'}$  is a 1-form with the values  $\text{End}\mathcal{V}$ , and the de Rham and Hodge bidegrees of the 1-form  $\nu$  coincide. Thus the Hodge bidegrees of  $\nu$  must be  $(1, 0)$  or  $(0, 1)$ , so it is real and closed. Thus recovered the conditions (106) on  $\nu$ .

The function  $G_{p, q}$  is zero unless  $p, q \geq 1$ . Indeed, otherwise the “ $(p-1, q-1)$ -rectangle” component of the Hodge  $(p, q)$ -part of the complex  $\mathcal{C}_{\mathcal{H}}^{\bullet}(\text{End}\mathcal{V})$  is zero. It must be imaginary thanks to the condition (74). So we recover conditions (105).

Vice versa, if these conditions are satisfied, we get a form (102). The lemma is proved.

Let us elaborate now the Maurer-Cartan equations (103). Specifying definitions from Section 4.3, consider the 1-form  $\psi := \mathbf{d}^c G + \nu$  on  $X(\mathbb{C})$ . Then  $\overline{G} = -G$  and  $\overline{\nu} = \nu$  imply  $\overline{\psi} = \psi$ . Denote by  $\psi_{p, q}^{a, b}$  the component of  $\psi$ , which is an  $(a, b)$ -form with values in  $\text{End}^{-p, -q}\mathcal{V}$ .

**Lemma 4.7** *The Maurer-Cartan equation (103) is equivalent to a system of differential equations*

$$(2p-2)\partial G_{p, q} = \sum_{p'+p''=p, \quad q'+q''=q} (p'+q'-1) \left[ G_{p', q'}, \psi_{p'', q''}^{1, 0} \right], \quad (107)$$

$$(2q-2)\overline{\partial} G_{p, q} = \sum_{p'+p''=p, \quad q'+q''=q} (p'+q'-1) \left[ G_{p', q'}, \psi_{p'', q''}^{0, 1} \right], \quad (108)$$

$$2\partial\overline{\partial} G_{1, 1} = [\nu, \nu]. \quad (109)$$

**Proof.** Follows from formulae (81), (84) for the product  $*$  and differential  $\delta$  in the Hodge complex. Observe that the differential  $\delta$  is the Laplacian only if in the right hand side stands an element  $*$  with  $N(*) = 0$ , which can happen only when  $(p, q) = (1, 1)$ . The lemma is proved.

**Remark.** The Maurer-Cartan equation (103) does not impose any conditions on  $\partial' G_{1, q}$  and  $\partial'' G_{p, 1}$  for  $p, q > 1$ . The complex conjugation interchanges differential equations (107) and (108).

Finally, the twistor connection is given explicitly by

$$\nabla_{\mathcal{G}} = \mathbf{d} + \sum_{p, q \geq 0} (1-u)^p (1+u)^q \left( (p+q+1) G_{p+1, q+1} du + \psi_{p+1, q}^{1, 0} + \psi_{p, q+1}^{0, 1} \right). \quad (110)$$

## 4.2 Proof of Theorem 4.5

Let us define a functor  $\mathcal{F} : \tilde{\mathcal{G}}_{\mathcal{H}}(X) \longrightarrow \text{MHod}_X$ . So let us construct, in a functorial way, a variation of mixed  $\mathbb{R}$ -Hodge structures on  $X(\mathbb{C})$  starting from a Green datum  $\mathcal{G} = \{G_{p,q}, \nu\}$  on a variation  $(\mathcal{V}, \mathbf{d})$  of real Hodge structures on  $X(\mathbb{C})$ .

(i) *The flat connection.* Restricting the flat connection  $\nabla_{\mathcal{G}}$  on  $p^*\mathcal{V}_{\infty}$  to  $X(\mathbb{C}) \times \{0\}$  we get a flat connection on the sheaf  $\mathcal{V}_{\infty}$  on  $X(\mathbb{C})$  given by

$$\nabla_{\mathcal{G}}^{(0)} := \mathbf{d} + \sum_{p,q \geq 0} (\psi_{p+1,q}^{1,0} + \psi_{p,q+1}^{0,1}). \quad (111)$$

(ii) *The weight filtration.* It is the standard weight filtration on a bigraded object:

$$W_n \mathcal{V}_{\mathbb{C}} := \oplus_{p+q \leq n} \mathcal{V}^{p,q}.$$

It is defined over  $\mathbb{R}$ . The connection  $\nabla_{\mathcal{G}}^{(0)}$  preserves the weight filtration.

(iii) *The Hodge filtration.* The standard Hodge filtration on a bigraded object provides the standard Hodge filtration on the restriction of  $p^*\mathcal{V}_{\infty}$  to  $X(\mathbb{C}) \times \{1\}$ :

$$F_{\text{st}}^p := \oplus_{i \geq p} \mathcal{V}^{i,*}. \quad (112)$$

Let  $P$  be the operator of parallel transport for the connection  $\nabla_{\mathcal{G}}$  along the twistor line from  $X \times \{1\}$  to  $X \times \{0\}$ . The Hodge filtration  $F^{\bullet}$  on  $\mathcal{V}_{\mathbb{C}}$  is the image of the standard one  $F_{\text{st}}^p$  by the operator  $P$ :

$$F^p \mathcal{V}_{\mathbb{C}} := P(F_{\text{st}}^p). \quad (113)$$

(iv) *The Griffiths transversality condition.* We employ the following general observation.

**Lemma 4.8** *Let  $(E, \nabla)$  be a smooth bundle with a flat connection on  $X(\mathbb{C}) \times \mathbb{R}$ , and  $\mathcal{F}^{\bullet}$  a filtration on  $E$  invariant under the  $\nabla$ -parallel transport along the twistor lines. Then if the restriction of  $(E, \mathcal{F}^{\bullet}, \nabla)$  to  $X(\mathbb{C}) \times \{s\}$  satisfies the Griffiths transversality condition for a single  $s \in \mathbb{R}$ , the same is true for any  $s$ .*

We apply Lemma 4.8 to the bundle  $E = p^*\mathcal{V}_{\infty}$  with the connection  $\nabla_{\mathcal{G}}$  and the filtration  $\mathcal{F}^{\bullet}$  given by the parallel transport of the standard Hodge filtration  $F_{\text{st}}^{\bullet}$  at  $u = 1$  along the twistor lines. The standard Hodge filtration (112) on the restriction of  $p^*\mathcal{V}_{\infty}$  to  $X(\mathbb{C}) \times \{1\}$  satisfies the Griffiths transversality condition. Indeed, the restriction of the connection  $\nabla_{\mathcal{G}}$  to  $X(\mathbb{C}) \times \{1\}$  is given by

$$\nabla_{\mathcal{G}}^{(1)} = \mathbf{d} + \nu^{0,1} + \nu^{1,0} + \sum_{q \geq 0} 2^q (\psi_{1,q}^{1,0} + \psi_{0,q+1}^{0,1}). \quad (114)$$

Therefore the Hodge filtration (113) at  $u = 0$  also satisfies the Griffiths transversality condition.

(v) *Mixed Hodge structure fiberwise.* Over a point Deligne's operator  $g$  can be recovered from  $G$ , and vice versa. Indeed, they are related by  $\exp(g) = P$ . The latter is given explicitly by

$$P = 1 + \sum_{n > 0} \left( \int_0^1 \varphi_{p_1, q_1} \circ \dots \circ \varphi_{p_n, q_n} \right) G_{p_1+1, q_1+1} \dots G_{p_n+1, q_n+1} \quad (115)$$

where  $\varphi_{p,q} := (p+q+1)(1-u)^p(1+u)^q du$ , and  $\int_0^1 \varphi_{p_1,q_1} \circ \dots \circ \varphi_{p_n,q_n}$  is the iterated integral of the 1-forms  $\varphi_{p_1,q_1}, \dots, \varphi_{p_n,q_n}$ . Therefore we have

$$g_{p,q} = (\log P)_{p,q} = \int_0^1 \varphi_{p-1,q-1} G_{p,q} + \dots$$

where  $\dots$  denotes a polynomial with rational coefficients of degrees  $\geq 2$  in  $G_{*,*}$ , of the total Hodge bidegree  $(p, q)$ . So by the induction  $G_{p,q}$  is recovered by a similar formulae from  $g_{*,*}$ .

We have defined the functor  $\mathcal{F}$ . It is clearly a tensor functor. To show that it induces an equivalence of categories we have to show that the functor  $\mathcal{F}$  is an isomorphism on Hom's, and every variation  $\mathcal{L}$  of mixed  $\mathbb{R}$ -Hodge structures can be obtained this way. The former statement is clear since it is known when  $X$  is point. So it remains to prove the latter statement.

Thanks to (v) for any variation  $\mathcal{L}$  of real MHS there is a unique smooth family of Green operators  $G_x$  on  $\text{gr}^W \mathcal{L}_x$ ,  $x \in X(\mathbb{C})$ , describing the real MHS's on the fibers  $\mathcal{L}_x$ , and there is a unique isomorphisms of fibrations  $\eta : \text{gr}^W \mathcal{L} \rightarrow \mathcal{L}$  which identifies, for every  $x$ , the real MHS provided by the pair  $(\text{gr}^W \mathcal{L}_x, G_x)$  with the real MHS  $\mathcal{L}_x$ . Let  $\mathbf{d} = \partial + \bar{\partial}$  be the connection on  $\text{gr}^W \mathcal{L}$  induced by the connection on  $\mathcal{L}$ . The isomorphism  $\eta^{-1}$  transforms the connection on  $\mathcal{L}$  to a connection  $\nabla$  on  $\text{gr}^W \mathcal{L}$ . We write it as  $\mathbf{d} + \mathcal{B}$ . We have to show that:

- (i) The connection on  $\mathcal{L}$  determines uniquely a real 1-form  $\nu$  as in (106).
- (ii) The Griffiths transversality condition is equivalent to the following two conditions:

*The operators  $G_x$  and the 1-form  $\nu$  from (i) satisfy differential equations (107)-(109). (116)*

$$\text{One has } \mathcal{B} = \mathbf{d}^c G + \nu. \quad (117)$$

Let  $\mathcal{B} = \mathcal{B}^{1,0} + \mathcal{B}^{0,1}$  be the decomposition into holomorphic and antiholomorphic components. Denote by  $\mathcal{B}_{p,q}$  (respectively  $\mathcal{B}_w$ ) the  $(p, q)$ - (respectively the  $w$ ) component of the operator valued 1-form  $\mathcal{B}$ ,<sup>5</sup> and similarly  $\mathcal{B}_{p,q}^{1,0}$  and  $\mathcal{B}_{p,q}^{0,1}$ .

Let  $\nu = \nu^{1,0} + \nu^{0,1}$  be the components of the connection  $\nabla$  of Hodge bidegrees  $(1, 0)$  and  $(0, 1)$ .

Let  $B = \{(p, q) | p, q \leq 0, p+q < 0\}$  be the domain in the  $(p, q)$ -plane shown on Fig. 7.

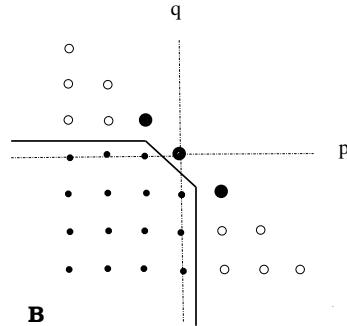


Figure 7: The components of the form  $\mathcal{B}$  can only be in the domain  $-B$ .

<sup>5</sup> $\mathcal{B}_{p,q}$  is the component of  $\mathcal{B}$  of the Hodge bidegree  $(-p, -q)$ . Its weight is  $-w$ .

**Lemma 4.9** (i)  $\mathcal{B}_{p,q} = 0$  unless  $(p, q)$  lies in the domain  $-B$  on Fig. 7.  
(ii) One has  $\mathcal{B}_1 = \nu$ .

**Proof.** Since the connection  $\nabla$  preserves the weight filtration, its components of negative weights are zero. The weight zero component of  $\nabla$  is the connection  $\mathbf{d}$  on  $\text{gr}^W \mathcal{L}$ .

The Griffiths transversality condition for the connection  $\mathbf{d} + \mathcal{B}$  just means that

$$(P^{-1}\partial P + P^{-1}\mathcal{B}^{1,0}P)_{p,q} = 0 \quad \text{if } p > 1. \quad (118)$$

$$(P^{-1}\bar{\partial} P + P^{-1}\mathcal{B}^{0,1}P)_{p,q} = 0 \quad \text{if } p > 0, \quad (119)$$

where  $P$  is the operator of parallel transform from 1 to 0 for the connection  $\nabla_{\mathcal{G}}$ . It is clear from (115) that:

a) The operators  $P$  and  $P^{-1}$  differ from the identity map by sums of terms of bidegrees  $\geq (1, 1)$ .

b) The components of  $P^{-1}\mathbf{d}P$  are inside of the domain  $-B$ .

This implies (i). The part (ii) follows from (i) and the very definition of  $\nu$ . The Lemma is proved.

Let us prove the claims (116)-(117) by the induction on  $w$ .

*The base of induction.* The  $w = 1$  case of (117) is given by the part (ii) of Lemma 4.9.

To get the rest, consider the following cousin of the twistor connection on  $X(\mathbb{C}) \times \mathbb{R}$ :

$$\nabla_{\mathcal{B},\mathcal{G}} = \mathbf{d} + \sum_{p,q \geq 0} (1-u)^p(1+u)^q \left( (p+q+1)G_{p+1,q+1}du + \mathcal{B}_{p+1,q}^{1,0} + \mathcal{B}_{p,q+1}^{0,1} \right). \quad (120)$$

Its curvature has two components: the horizontal one, and the one containing  $du$ , called the  $du$ -part.

**Lemma 4.10** *The  $du$ -part of curvature of  $\nabla_{\mathcal{B},\mathcal{G}}$  is zero if and only if the Griffiths transversality condition for the connection  $\mathbf{d} + \mathcal{B}$  holds.*

**Proof.** Since the  $du$ -components of the connections  $\nabla_{\mathcal{B},\mathcal{G}}$  and  $\nabla_{\mathcal{G}}$  are the same, the operator  $P$  is also the operator of parallel transform from 1 to 0 for the connection  $\nabla_{\mathcal{B},\mathcal{G}}$ . Further, the connection  $\mathbf{d} + \mathcal{B}$  coincides with the restriction of  $\nabla_{\mathcal{B},\mathcal{G}}$  to  $X \times \{0\}$ . The value of  $P^{-1}(\mathbf{d} + \mathcal{B})P$  on a tangent vector  $v$  at a point  $(x, 1)$  is a linear endomorphism of the fiber of  $p^*\mathcal{L}$  at  $(x, 1)$ . It follows that it is given by the composition of the parallel transform from 1 to 0, followed by the infinitesimal parallel transform along  $v$ , and then by the parallel transform back from 0 to 1 for the connection  $\nabla_{\mathcal{B},\mathcal{G}}$ . So it is the parallel transport for the connection  $\nabla_{\mathcal{B},\mathcal{G}}$  along the three sides of an infinitesimal rectangle  $R$  with the sides  $v$  and  $x \times [0, 1]$ . Thus it differs from the value of the connection  $\nabla_{\mathcal{B},\mathcal{G}}$  on  $v$  by the integral of the curvature over  $R$ . The restriction of the connection  $\nabla_{\mathcal{B},\mathcal{G}}$  to  $X \times \{1\}$  is given by

$$\nabla_{\mathcal{B},\mathcal{G}}^{(1)} = \mathbf{d} + \sum_{q \geq 0} 2^q (\mathcal{B}_{1,q}^{1,0} + \mathcal{B}_{0,q+1}^{0,1}).$$

Therefore it satisfies the Griffith transversality condition. The  $(p, q)$ -component of the latter equals  $\beta_{p,q} := \int_0^1 (1-u)^p(1+u)^q du$  times the  $(p, q)$ -component of (the  $du$ -part of the curvature)/ $du$ . Notice that  $\beta_{p,q} > 0$  – we integrate a positive function. The lemma is proved.

Thanks to the Griffiths transversality for the connection  $\mathbf{d} + \mathcal{B}$  and Lemma 4.10, we have

$$P^{-1}\partial P + P^{-1}\mathcal{B}^{1,0}P = \nabla_{\mathcal{B},\mathcal{G}}^{1,0}|_{u=1} = \sum_{q \geq 0} 2^q \mathcal{B}_{1,q}^{1,0}. \quad (121)$$

$$P^{-1}\bar{\partial}P + P^{-1}\mathcal{B}^{0,1}P = \nabla_{\mathcal{B},\mathcal{G}}^{0,1}|_{u=1} = \sum_{q \geq 0} 2^q \mathcal{B}_{0,q+1}^{0,1}. \quad (122)$$

*The induction step.* Let us assume the claims for  $w \leq n-1$ . The right hand sides of the weight  $n$  Maurer-Cartan equations (107)-(109) involve only  $\nu$  and functions  $G$  with  $p+q < n$ . Therefore using Lemma 4.10 plus the induction assumption we see that the Griffiths transversality for the connection  $\mathbf{d} + \mathcal{B}$  implies that  $G_{p,q}$  with  $p+q = n$  satisfy differential equations (107)-(109).

The weight  $n$  parts of equations (121)-(122) look as follows:

$$\mathcal{B}_n^{1,0} + 2^{n-1}\mathcal{B}_{1,n-1}^{1,0} = \Phi_1^{(n)}(G, \nu), \quad \mathcal{B}_n^{0,1} + 2^{n-1}\mathcal{B}_{0,n}^{0,1} = \Phi_2^{(n)}(G, \nu). \quad (123)$$

Here  $\Phi_*^{(n)}(G, \nu)$  are differential polynomials of  $\nu$  and  $G_{p,q}$ ,  $(p+q) \leq n$ . Indeed, by the induction assumption  $\mathcal{B}_w$ ,  $w \leq n-1$ , have this property, and by (115)  $P$  is a differential polynomial in  $G$ 's.

We have proved that  $G_{p,q}$ ,  $p+q = n$  satisfy differential equations (107)-(109). Therefore thanks to Proposition 4.3 (where we put  $G_{p,q} = 0$  if  $p+q > n$ ) there is a solution of (123) with the same  $\nu$  and  $G$ 's, and with  $\mathcal{B}_n = \mathbf{d}^c G_n$ . Since  $\mathcal{B}_n$  is determined from (123), this implies that  $\mathcal{B}_n = \mathbf{d}^c G_n$ . So we get (117) for  $w = n$ . The theorem is proved.

**Example.** Solving equations (121)-(122) for the components of  $\mathcal{B}_2$ , we have

$$\lambda^{1,0} \circ G_{1,1} + \mathcal{B}_{2,0}^{1,0} = 0, \quad \bar{\partial}_0 G_{1,1} + \mathcal{B}_{1,1}^{0,1} = 0, \quad \mathcal{B}_{2,0}^{0,1} = 0.$$

Applying the complex conjugation, we get

$$\lambda^{0,1} \circ G_{1,1} = \mathcal{B}_{0,2}^{0,1}, \quad \partial_0 G_{1,1} = \mathcal{B}_{1,1}^{1,0}, \quad \mathcal{B}_{0,2}^{1,0} = 0.$$

Thus

$$\mathcal{B}_2 = \partial_0 G_{1,1} - \lambda^{1,0} \circ G_{1,1} - \bar{\partial}_0 G_{1,1} + \lambda^{0,1} \circ G_{1,1} = \mathbf{d}^c G_{1,1}.$$

### 4.3 DG generalizations

Let us recall some well known definitions. A DG Lie algebra is a graded Lie algebra with a degree  $+1$  differential satisfying the Leibniz rule. A DG-module over a DG Lie algebra is a module over a Lie algebra in the category of complexes rather than vector spaces. So a DG module over a DG Lie algebra  $(L_\bullet, \partial)$  is a complex  $M_\bullet$  plus an action of the graded Lie algebra  $L_\bullet$  on the graded object  $M_\bullet$  given by a map of complexes  $\mu : L_\bullet \otimes M_\bullet \rightarrow M_\bullet$ .

Given two DG-modules  $M_\bullet$  and  $N_\bullet$ ,  $\text{Hom}_{L_\bullet}^\bullet(M_\bullet, N_\bullet)$  is a complex, consisting of elements of the complex  $\text{Hom}^\bullet(M_\bullet, N_\bullet)$  commuting with the action of  $L_\bullet$ . Taking  $H^0$ 's of these Hom-complexes we arrive at the homotopy category of DG-modules. Here the morphisms are morphisms of  $L_\bullet$ -modules  $M_\bullet \rightarrow N_\bullet$  commuting with the differentials in  $M_\bullet$  and  $N_\bullet$ , up to homotopies given by morphisms of  $L_\bullet$ -modules  $M_\bullet \rightarrow N_\bullet[-1]$ . Localizing by the quasiisomorphisms, we get the derived category  $\mathcal{D}(L_\bullet)$ . One easily translates this to the world of DG Lie coalgebras.

Recall the DG Lie coalgebra  $\mathcal{L}_{\mathcal{H};X}^*$  in the pure category  $\text{Hod}_X$  defined in Section 1.8.



**Definition 4.11** *The DG Hodge category  $\mathcal{G}_{\mathcal{H}}^*(X)$  of a regular complex projective variety  $X$  is the derived category  $\mathcal{D}(\mathcal{L}_{\mathcal{H};X}^*)$  of DG comodules over the DG coalgebra  $\mathcal{L}_{\mathcal{H};X}^*$  in the pure category  $\text{Hod}_X$ .*

The precise form of Conjecture 1.8 is

**Conjecture 4.12** *The category of smooth complexes of real Hodge sheaves on  $X$  is equivalent to the derived category  $\mathcal{D}(\mathcal{L}_{\mathcal{H};X}^*)$ .*

Let us describe DG comodules over the DG Lie coalgebra  $\mathcal{L}_{\mathcal{H};X}^*$ . Consider the following data:

- A bounded complex  $\mathcal{V} = \{\dots \rightarrow \mathcal{V}^1 \xrightarrow{\partial} \mathcal{V}^2 \xrightarrow{\partial} \mathcal{V}^3 \xrightarrow{\partial} \mathcal{V}^4 \xrightarrow{\partial} \dots\}$  in  $\text{Hod}_X$ .
- A Maurer-Cartan element:

$$\mathcal{G} \in \mathcal{C}_{\mathcal{H}}^1(\text{End}\mathcal{V}) \quad \text{such that} \quad \delta\mathcal{G} + [\mathcal{G}, \mathcal{G}] = 0, \quad (124)$$

defined modulo elements of type  $\delta B + [B, \mathcal{G}]$ , where  $B \in \mathcal{C}_{\mathcal{H}}^0(\text{End}\mathcal{V})$ .

Here  $\text{End}\mathcal{V} = \mathcal{V}^\vee \otimes \mathcal{V}$  is a complex in  $\text{Hod}_X$ . The complex  $\mathcal{C}_{\mathcal{H}}^*(\text{End}\mathcal{V})$  is the total complex of the bicomplex  $\mathcal{C}_{\mathcal{H}}^\bullet(\text{End}\mathcal{V})$ . So an element (124) includes a function on  $X(\mathbb{C})$  with values in  $\text{End}^0\mathcal{V}$ , a 1-form with values in  $\text{End}^{-1}\mathcal{V}$ , etc. Thanks to Theorem 4.5 it provides the cohomology  $H^*(\mathcal{V})$  with a variation of  $\mathbb{R}$ -MHS's data.

**Lemma 4.13** (i) *DG comodules over the DG Lie coalgebra  $\mathcal{C}_{\mathcal{H}}^*(\text{End}\mathcal{V})$  are the same thing as Maurer-Cartan elements (124).*

(ii) *Homotopy equivalent DG comodules correspond to the equivalent Maurer-Cartan elements.*

**Proof.** (i) Neglecting the differential, the graded Lie coalgebra  $\mathcal{L}_{\mathcal{H};X}^*$  is a free Lie coalgebra cogenerated by the graded object  $\mathcal{D}_X[1]$  in the category  $\text{Hod}_X$ . Therefore a comodule over  $\mathcal{L}_{\mathcal{H};X}^*$  is determined by the coaction of the cogenerators. The latter is given by a graded object  $\mathcal{V}$  plus a linear map  $\mathcal{V} \rightarrow \mathcal{V} \otimes_{\mathbb{R}} \mathcal{D}_X[1]$ . It can be interpreted as a degree zero element in  $\mathcal{D}_X[1] \otimes_{\mathbb{R}} \text{End}\mathcal{V}$ , which is the same as an element  $\mathcal{G} \in \mathcal{C}_{\mathcal{H}}^1(\text{End}\mathcal{V})$ .

ii) Similarly a homotopy is given by an element  $\mathcal{V} \rightarrow \mathcal{V} \otimes_{\mathbb{R}} \mathcal{D}_X$ , which can be interpreted as an element  $B \in \mathcal{C}_{\mathcal{H}}^0(\text{End}\mathcal{V})$ .

Assume that  $L_k = 0$  for  $k > 0$ . Then subcategory of DG-modules concentrated in the degree zero is canonically identified with the category of modules over the Lie algebra  $H_0(L_\bullet)$ . Indeed, given such a module  $M_0$ ,  $L_k$  where  $k \neq 0$  acts on it trivially since  $M_0$  a graded module over a graded Lie algebra. Further, since the action map  $\mu$  is a map of complexes,  $\partial(L_{-1})$  acts on  $M_0$  trivially.

There is a similar picture for DG comodules over a DG Lie coalgebra concentrated in degrees  $\geq 0$ , which we use below.

**Proof of Lemma 4.4.** Follows immediately from Lemma 4.13.

A *DG-connection* on  $X(\mathbb{C})$  is a first order linear differential operator  $\mathbf{d} + \mathcal{A}$  where  $\mathcal{A}$  is an  $\text{End}\mathcal{V}$ -valued differential form on  $X(\mathbb{C})$  of total degree 1. A DG-connection is *flat* if  $(\mathbf{d} + \mathcal{A})^2 = 0$ .

**Definition 4.14** Let  $\mathcal{V}$  be a variation of real Hodge structures on  $X(\mathbb{C})$ , and  $\mathcal{G} \in \mathcal{C}_{\mathcal{H}}^1(\text{End}\mathcal{V})$ . The twistor DG-connection of  $\mathcal{G}$  is a DG-connection on  $X(\mathbb{C})$  given by  $\nabla_{\mathcal{G}} := \mathbf{d} + \gamma(\mathcal{G})$ .

**Proposition 4.15** A form  $\mathcal{G} \in \mathcal{C}_{\mathcal{H}}^1(\text{End}\mathcal{V})$  satisfies the Maurer-Cartan equation  $\delta\mathcal{G} + [\mathcal{G}, \mathcal{G}] = 0$  if and only if the twistor DG-connection  $\nabla_{\mathcal{G}}$  is flat.

**Proof.** Follows immediately from Theorem 1.6.

## 5 Plane trees and DG Lie coalgebras

Below we use a set-up which we see again in Sections 6-10. Let  $H^{\vee}$  be a finite dimensional symplectic vector space, and  $S$  a finite set. Set

$$V_{H,S}^{\vee} := H^{\vee} \oplus \mathbb{C}[S].$$

If  $X$  is a smooth compact complex curve,  $H^{\vee} = H^1(X, \mathbb{C})$ , and  $S \subset X$ , this is the space  $V_{X,S}^{\vee}$  from Sections 1 and 2. Since  $H$  never denotes a curve, this does not lead to a confusion. We denote vectors in  $H$  by Greek letters.

Let  $\mathcal{T}_{H,S}^{\vee, \bullet}$  be the graded vector space spanned by plane trees decorated by  $V_{H,S}^{\vee}$ . The grading of a generator assigned to a plane tree  $T$  is  $1 + \sum(\text{val}(v) - 3)$  where  $v$  runs through the set of all internal vertices of  $T$ . We introduce a DG Lie coalgebra structure on  $\mathcal{T}_{H,S}^{\vee, \bullet}[1]$ . It generalizes both the construction in [G5] and the Kontsevich-Boardman differential in the graph complex.

The standard cochain complex of the DG Lie coalgebra  $\mathcal{T}_{H,S}^{\vee, \bullet}[1]$  can be described as a commutative DG algebra  $\mathcal{F}_{H,S}^{\vee, \bullet}$  spanned by plane forests decorated by  $V_{H,S}^{\vee}$ .

Let  $\mathcal{C}_{H,S}^{\vee} := \text{CT}(V_{H,S}^{\vee})$  be the cyclic envelope of the tensor algebra of  $V_{H,S}^{\vee}$ . We define a Lie coalgebra structure on  $\mathcal{C}_{H,S}^{\vee}$ . The coproduct generalizes the coproduct for the dihedral Lie coalgebras from [G4] – the latter corresponds to the  $H^{\vee} = 0$  case.

The sum over all plane trivalent trees with a given decoration gives rise to an injective map

$$F : \mathcal{C}_{H,S}^{\vee} \hookrightarrow \mathcal{T}_{H,S}^{\vee, 1}[1].$$

We show that it is a morphism of Lie coalgebras. Therefore we arrive at an injective map of the corresponding standard (Chevalier) cochain complexes

$$F^{\bullet} : S^{\bullet}(\mathcal{C}_{H,S}^{\vee}[-1]) \hookrightarrow \mathcal{F}_{H,S}^{\vee, \bullet}. \quad (125)$$

Notice that  $S^{\bullet}(\mathcal{C}_{H,S}^{\vee}[-1]) = \Lambda^{\bullet}(\mathcal{C}_{H,S}^{\vee})$ .

The Lie coalgebra structure on  $\mathcal{C}_{H,S}^{\vee}$  provides the dual vector space  $\mathcal{C}_{H,S}$  with a Lie algebra structure. We give a different interpretation of the Lie algebra  $\mathcal{C}_{H,S}$  in Sections 7.

### 5.1 The DG Lie coalgebra of plane decorated trees

Below a plane tree is a tree with internal vertices of valency  $\geq 3$ . A tree may have no internal vertices, i.e. just two external vertices. A *plane forest* is a disjoint union of plane trees. An *R*-decorated plane forest is a plane forest decorated by a set  $R$ . If  $F = T_1 \cup \dots \cup T_k$  is a forest presented as a union of trees  $T_i$ , its orientation torsor is the product of the tree orientation torsors:  $\text{or}_F = \text{or}_{T_1} \otimes \dots \otimes \text{or}_{T_k}$ .

*Decorations by cyclic words.* A decoration of a tree  $T$  by vectors of a vector space  $V$  gives rise to a decoration of  $T$  by an element  $\mathcal{CT}(V)$ : if the external edges are decorated clockwise by the vectors  $v_1, \dots, v_n$ , the resulting decoration is a cyclic word  $W := \mathcal{C}(v_1 \otimes \dots \otimes v_n)$ . We say then that  $T$  is decorated by the cyclic word  $W$ . Different decorations of external edges of  $T$  by vectors  $v_i$  can lead to the same decoration by a cyclic word  $W$ . A decoration of a forest is given by decorations  $W_{T_i}$  of its connected components  $T_i$ . All constructions below depend only on the symmetric product  $W_{T_1} \circ \dots \circ W_{T_n}$  of cyclic words.

**The commutative DGA  $\mathcal{F}_{H,S}^{\vee, \bullet}$ .** The graded vector space  $\mathcal{F}_{H,S}^{\vee, \bullet}$  is generated by oriented forests decorated by tensor products of cyclic words in  $V_{H,S}^{\vee}$ . The grading is given by the degree

$$\deg(F) := \sum_{\text{vertices } v \text{ of } F} (\text{val}(v) - 3) + \pi_0(F).$$

There is a structure of a graded commutative algebra on  $\mathcal{F}_{H,S}^{\vee, \bullet}$ , defined on the generators by

$$(F_1, W_1; \text{Or}_{T_1}) * (F_2, W_2; \text{Or}_{T_2}) = (F_1 \cup F_2, W_1 \circ W_2; \text{Or}_{T_1} \wedge \text{Or}_{T_2}).$$

Let us define a differential  $\partial : \mathcal{F}_{H,S}^{\vee, \bullet} \rightarrow \mathcal{F}_{H,S}^{\vee, \bullet+1}$ . When  $H^{\vee} = 0$  it is the differential defined in [G5]. We define it on trees, and then extend by the Leibniz rule. It has three components:

$$\partial = \partial_{\Delta} + \partial_{\text{Cas}} + \partial_S.$$

(i) *The map  $\partial_{\Delta}$ .* Let  $(T, W_T; \text{Or}_T)$  be a generator and  $E$  an *internal* edge of  $T$ , see Fig 8. Let  $T/E$  be the tree obtained by contraction of the edge  $E$ . So it has one less edge, and one less vertex. The orientation  $\text{Or}_T$  of the tree  $T$  induces an orientation  $\text{Or}_{T/E}$  of the tree  $T/E$ . Namely, if  $\text{Or}_T = E \wedge E_1 \wedge E_2 \wedge \dots$  then  $\text{Or}_{T/E} = E_1 \wedge E_2 \wedge \dots$ . Contracting an internal edge we do not touch the decoration. Set

$$\partial_{\Delta}(T, W_T; \text{Or}_T) := \sum_{E: \text{ internal edges of } T} (T/E, W_T; \text{Or}_{T/E}).$$

It is (a decorated version of) the graph complex differential of Boardman and Kontsevich.

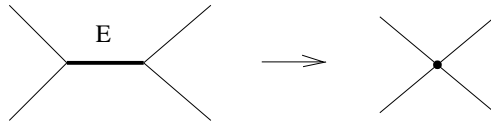


Figure 8:  $\partial_{\Delta}$ : The  $\delta_X$ -term for an internal edge  $E$ .

(ii) *The map  $\partial_{\text{Cas}}$ .* Let  $E$  be an edge of  $T$ . Cut the tree  $T$  along the edge  $E$ , getting two trees,  $T_1$  and  $T_2$ , see Fig 9 and Fig 10. Choose their orientations  $\text{Or}_{T_1}$  and  $\text{Or}_{T_2}$  so that  $\text{Or}_T = E \wedge \text{Or}_{T_1} \wedge \text{Or}_{T_2}$ . The trees  $T_i$  inherit partial decorations by the (non-cyclic!) words  $W'_i$ , so that  $W_T = \mathcal{C}(W'_1 \otimes W'_2)$ . Denote by  $E_1$  and  $E_2$  the new external edges of the trees  $T_1$  and  $T_2$  obtained by cutting  $E$ .

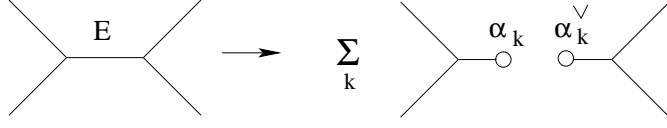


Figure 9:  $\partial_{\text{Cas}}$ : The Casimir term for an internal edge  $E$ .

Below we use the identity (Casimir) element  $\text{Id} : H \rightarrow H$ . We write it as follows. Choose a basis  $\{\alpha_k\}$  of  $H$ . Denote by  $\{\alpha_k^\vee\}$  the dual basis:  $(\alpha_k, \alpha_l^\vee) = \delta_{kl}$ . Then  $\text{Id} = \sum_k \alpha_k^\vee \otimes \alpha_k$ .

We decorate  $E_1$  by  $\alpha_k$ ,  $E_2$  by  $\alpha_k^\vee$ , getting a decoration  $\mathcal{C}(W_1 \otimes \alpha_k)$  of the tree  $T_1$ , and a decoration  $\mathcal{C}(\alpha_k^\vee \otimes W_2)$  of the tree  $T_2$ . Then

$$\partial_{\text{Cas}}(T, W_T; \text{Or}_T) := \sum_E \sum_k (T_1, \mathcal{C}(W_1' \otimes \alpha_k); \text{Or}_{T_1}) \wedge (T_2, \mathcal{C}(\alpha_k^\vee \otimes W_2'); \text{Or}_{T_2}) \quad (126)$$

where the first sum is over all internal edges  $E$  of  $T$ .

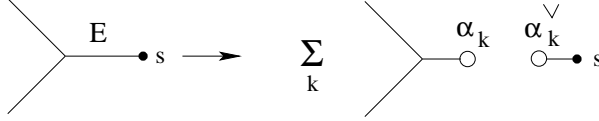


Figure 10:  $\partial_{\text{Cas}}$ : Contribution of the Casimir term for an external edge  $E$ .

(iii) *The map  $\partial_S$ .* Let  $E$  be an external  $S$ -decorated edge of  $T$ , see Fig 11. Remove it together with a little neighborhood of its vertices. If  $E \neq T$ , one of the vertices has the valency  $v \geq 3$ . The tree  $T$  is replaced by  $v - 1$  trees  $T_1, \dots, T_{v-1}$ : each of the internal edges sharing the vertex with  $E$  produces a new tree. Choose their orientations  $\text{Or}_{T_i}$  so that  $\text{Or}_T = E \wedge \text{Or}_{T_1} \wedge \dots \wedge \text{Or}_{T_{v-1}}$ . The decoration of the tree  $T$  provides a decorations  $W_{T_i}$  of the new trees  $T_i$ , so that the new

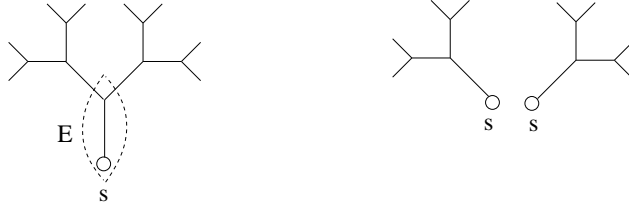


Figure 11:  $\partial_S$ : Contribution of the  $\delta_X$ -term for an external edge  $E$ .

external vertex of each of the trees  $T_i$  inherits the  $S$ -decoration of  $E$ . Set

$$\frac{\partial}{\partial E}(T, W_T; \text{Or}_T) = (T_1, W_{T_1}; \text{Or}_{T_1}) \wedge \dots \wedge (T_{v-1}, W_{T_{v-1}}; \text{Or}_{T_{v-1}}), \quad \partial_S := \sum_{E: S\text{-decorated}} \frac{\partial}{\partial E}.$$

**Proposition 5.1** *One has  $\partial^2 = 0$ . So  $\mathcal{F}_{H,S}^{\vee, \bullet}$  is a commutative DGA.*

**Proof.** It is well known that  $\partial_\Delta^2 = 0$ . To check that  $\partial_{\text{Cas}}^2 = 0$  observe that  $\partial_{\text{Cas}}^2(W)$  is a sum of two terms, one of whom is

$$\sum_{E_1, E_2} \sum_{k, l} (T_1, \mathcal{C}(W_1 \otimes \alpha_k); \text{Or}_{T_1}) \wedge (T_2, \mathcal{C}(\alpha_k^\vee \otimes W_2 \otimes \beta_l); \text{Or}_{T_2}) \wedge (T_3, \mathcal{C}(\beta_l^\vee \otimes W_3); \text{Or}_{T_3}) \quad (127)$$

and the other has the opposite sign, see the left picture on Fig 12.

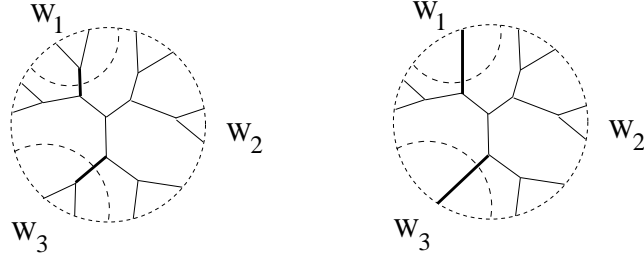


Figure 12:  $\partial_{\text{Cas}}^2 = 0$  and  $\partial_S^2 = 0$ .

There we cut the tree  $T$  along the edges  $E_1$  and  $E_2$ , shown by the thick segments. The dotted arcs show how cutting along these edges we get three new trees. A similar argument shows that  $\partial_S^2 = 0$ , see the right picture on Fig 12. Computing the signs we see that the differentials  $\partial_\Delta, \partial_{\text{Cas}}, \partial_S$  anticommute. The Proposition is proved.

## 5.2 A Lie coalgebra structure on $\mathcal{C}_{H,S}^\vee$

The cobracket  $\delta : \mathcal{C}_{H,S}^\vee \longrightarrow \Lambda^2 \mathcal{C}_{H,S}^\vee$  is defined as a sum  $\delta = \delta_{\text{Cas}} + \delta_S$ . Let us define these maps. We picture a cyclic word  $W$  on an oriented circle.

(i) *The map  $\delta_{\text{Cas}}$ .* Cut two different arcs of the circle. We get two semicircles. Make an oriented circle out of each of them. Each of them comes with a special point on it, obtained by gluing the ends of the semicircle. Decorate the special point of the first circle by  $\alpha_k$ , and the special point of the second circle by  $\alpha_k^\vee$ , take the sum over  $k$ , and put the sign. This is illustrated on Fig 13. The map  $\delta_{\text{Cas}}$  is obtained by taking the sum over all possible cuts.

Observe that the wedge product comes with the plus sign, and changing the order of the cuts (and hence the order of the terms in the wedge product) we do not change it, since if  $\{\alpha_k\}$ ,  $\{\alpha_k^\vee\}$  is a basis and its symplectic dual, then the same is true for  $\{-\alpha_k^\vee\}$ ,  $\{\alpha_k\}$ .

(ii) *The map  $\delta_S$ .* Cut the circle at an arc and at an  $S$ -decorated point, which is not at the end of the arc, and make two circles as before. Their special points inherit an  $S$ -decoration from the  $S$ -decorated point, see Fig 13. Make the sum over all possible cuts.

This time the order of the terms in the wedge product matters. We put the semicircle which, going according to the orientation of the circle, has the  $S$ -cut at the end as the first term.

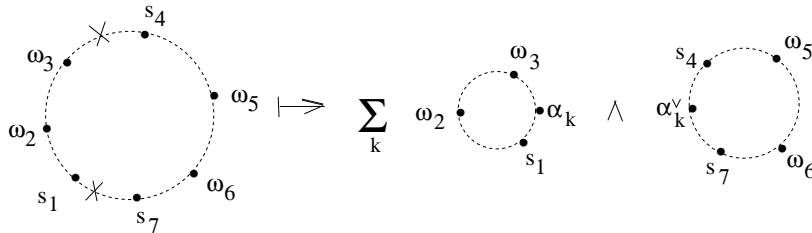


Figure 13: The component  $\delta_{\text{Cas}}$  of the differential.

**Examples.** One has

$$\delta \mathcal{C}(\{s_0\} \otimes \{s_1\}) = \mathcal{C}(\{s_0\} \otimes \alpha_k) \wedge \mathcal{C}(\alpha_k^\vee \otimes \{s_1\}). \quad (128)$$

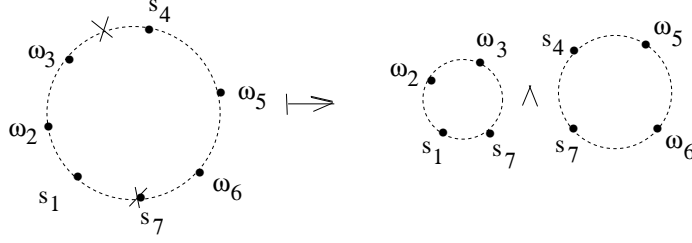


Figure 14: The component  $\delta_S$  of the differential.

$$\delta \mathcal{C}(\{s_0\} \otimes \{s_1\} \otimes \{s_2\}) = \text{Cycle}_{0,1,2} \left( \mathcal{C}(\{s_0\} \otimes \{s_1\} \otimes \alpha_k) \wedge \mathcal{C}(\alpha_k^\vee \otimes \{s_2\}) + \mathcal{C}(\{s_0\} \otimes \{s_1\}) \wedge \mathcal{C}(\{s_1\} \otimes \{s_2\}) \right). \quad (129)$$

where  $\text{Cycle}_{0,1,2}$  means the cyclic sum. See Fig 15 illustrating the last formula.

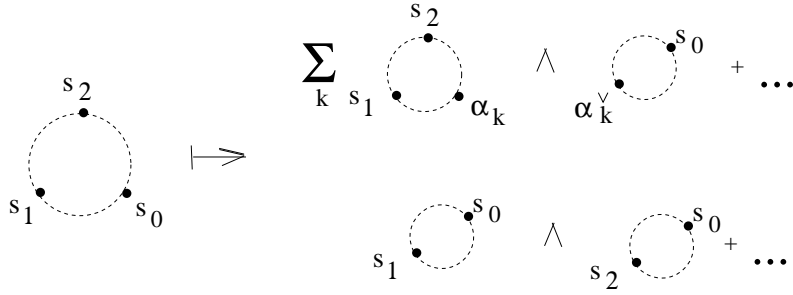


Figure 15: The coproduct of  $\mathcal{C}(\{s_0\} \otimes \{s_1\} \otimes \{s_2\})$ .

**Proposition 5.2** *One has  $\delta^2 = 0$ .*

**Proof.** Let us define an injective linear map

$$F : \mathcal{C}_{H,S}^\vee \hookrightarrow \mathcal{F}_{H,S}^{\vee,1}[1], \quad F(W) := \sum_T (T, W; \text{Or}_T).$$

Here the sum is over all plane trivalent trees decorated by the cyclic word  $W$ , and  $\text{Or}_T$  is the canonical orientation of the trivalent tree  $T$  corresponding to the clockwise orientation of the plane. We extended it to the map  $F^\bullet$ , see (125).

**Lemma 5.3** *The map  $F^\bullet$  commutes with the cobrackets:  $\partial F^\bullet = F^\bullet \delta$ .*

**Proof.** Let  $E$  be an internal edge. There are exactly two contributions to  $\partial_\Delta$  corresponding to the trees shown on Fig 16 (the parts of these graphs which are not shown are the same). They cancel each other since the corresponding trees with one 4-valent vertex obtained by shrinking the edge  $E$  inherit different orientations. Thus  $\partial_\Delta$  is zero on the image of the map  $F$ .

Let us check that the differentials  $\partial_{\text{Cas}}$  and  $\partial_S$  on  $\mathcal{T}_{H,S}^{\vee,\bullet}$  match the maps  $\delta_{\text{Cas}}$  and  $\delta_S$  for  $\mathcal{C}_{H,S}^\vee$ . Look at Fig 17. To show that  $\partial_{\text{Cas}}$  matches  $\delta_{\text{Cas}}$  take a cut shown by a punctured arc on the left, and vary the trees in the two obtained domains keeping their external vertices untouched. For the second pair of the differentials use the same argument for the cut shown on the right picture. The Lemma is proved. Proposition 5.2 follows from Lemma 5.3 and Proposition 5.1.

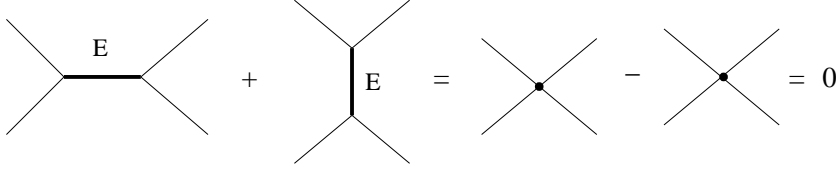


Figure 16: The two terms for the internal edges  $E$  cancel each other.

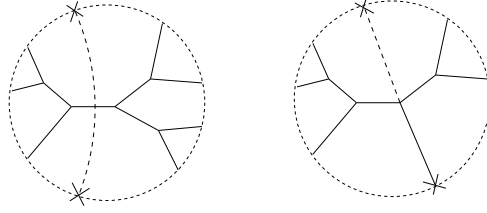


Figure 17: Compatibility of the differentials on forests and cyclic words.

**Lemma 5.4** a) One has  $\mathcal{C}_{H,S}^\vee = H_{\partial_\Delta}^0(\mathcal{T}_{H,S}^{\vee,\bullet}[1])$ .

b) The Lie coalgebra  $\mathcal{C}_{H,S}^\vee$  is quasiisomorphic to the DG Lie coalgebra  $\mathcal{T}_{H,S}^{\vee,\bullet}[1]$ . So the latter is a resolution of the former.

**Proof.** The part a) follows from the exactness of (70) in the second term. The part b) follows from Lemma 5.3.

## 6 Differential equations for Hodge correlators on curves

### 6.1 Hodge correlators over a base and Maurer-Cartan equations

*De Rham description of the The Gauss-Manin connection.* Let  $\pi : X \rightarrow B$  be a smooth map. Then the Gauss-Manin connection  $\nabla_{GM}$  in  $R^k\pi_*\mathbb{C}$  can be described as follows. Take a section  $s$  of  $R^k\pi_*\mathbb{C}$ . Choose a smooth family of differential forms  $\alpha_t$  on the fibers  $X_t(\mathbb{C})$  whose cohomology classes form the section  $s$ . Choose any differential form  $\beta$  on  $X(\mathbb{C})$  whose restriction to every fiber  $X_t(\mathbb{C})$  coincides with the form  $\alpha_t$ . Then  $d\beta$  gives rise to  $\nabla_{GM}s$  as follows. The de Rham differential of  $\beta$  along the fibers of  $\pi$  is zero since  $d\alpha_t = 0$  for every  $t$ . Therefore  $d\beta$  provides a 1-form on the base with values in  $\mathcal{A}_{cl}^{k-1}(X/B)$ . The cohomology class of the latter does not depend on the choice of the differential form  $\beta$  representing the section  $s$ . The obtained 1-form on  $B(\mathbb{C})$  with values in  $R^{k-1}\pi_*\mathbb{C}$  equals  $\nabla_{GM}(s)$ .

Let  $p : X \rightarrow B$  be a smooth family of complex projective curves over  $B$ , and  $S \subset X$  a smooth divisor in  $X$  over  $B$ . Let  $s_0$  be a single component of  $S$ , and  $S^* = S - s_0$ . Then  $\mathcal{C}_{X,S^*}$  and  $\mathcal{C}_{X,S^*}^\vee$  are variations of real Hodge structures over  $B(\mathbb{C})$ . Choose a section  $v_0$  of the fibration  $T_{s_0}X/B$ .

Let us define a linear map, the Hodge correlator map:

$$\text{Cor}_{\mathcal{H},v_0} : \mathcal{F}_{X,S^*}^{\vee,k} \longrightarrow \mathcal{D}_{B(\mathbb{C})}^{k-1}. \quad (130)$$

Below we often suppress  $v_0$  from the notation. Here  $\mathcal{D}_{B(\mathbb{C})}^k$  is the space of  $k$ -currents on  $B(\mathbb{C})$ . There is a canonical projection

$$p_F : X/B(\mathbb{C})^{\{\text{vertices of } F\}} \longrightarrow B(\mathbb{C}).$$

Here on the left stands the fibered product of copies of  $X$  over the base  $B$ . Similarly to the procedure in Section 2.3, let us assign to a generator  $(F, W; \text{Or}_F)$  given by a  $W$ -decorated oriented forest  $F$  a current  $\kappa(F, W; \text{Or}_F)$  on  $X_{/B}(\mathbb{C})\{\text{vertices of } F\}$ .

Write  $W$  as a product of  $k$  cyclic words  $W = W_1 \circ \dots \circ W_k$ . Let  $W_s = \mathcal{C}(w_0 \otimes \dots \otimes w_m)$ . Choose a form  $\beta_i$  on  $X(\mathbb{C})$  representing the section  $w_i$ : its restrictions to the fibers  $X_t$  are holomorphic/antiholomorphic forms representing the section  $w_i$ . Using the forms  $\beta_i$ , the Green functions assigned to the edges of the forest  $F$ , and the orientation  $\text{Or}_F$  to cook up a current  $\kappa(F, W; \text{Or}_F)$ . Then set

$$\text{Cor}_{\mathcal{H}}(F, W; \text{Or}_F) := p_{F*} \left( \kappa(F, W; \text{Or}_F) \right).$$

It does not depend on the choice of the forms  $\beta_i$ . Indeed, if  $\tilde{\beta}_i$  is a different choice, then the restriction of  $\tilde{\beta}_i - \beta_i$  to the fiber curve is a closed harmonic form there, and hence is zero. Here we used in a crucial way the fact that the fibers are smooth complex projective curves, and so there are canonical harmonic representatives for  $H^1$  given by holomorphic/antiholomorphic forms. When  $k = 1$  it is the defined earlier Hodge correlator map.

The Hodge correlator map (130) is  $\pi^* C_{B(\mathbb{C})}^\infty$ -linear by construction. Therefore, dualising, it can be viewed as an element

$$\tilde{\mathbf{G}} \in \mathcal{D}^{*,*}(\mathcal{T}_{X,S^*}) = \mathcal{D}_{B(\mathbb{C})}^{*,*} \otimes_{C_{B(\mathbb{C})}^\infty} \mathcal{T}_{X,S^*}.$$

Denote by  $\mathbb{F}_{S^*}^\bullet$  the graded commutative algebra of  $S^*$ -decorated forests. It is spanned by  $(F, \text{Or}_F)$ , where  $F$  is a forests decorated by components of the divisor  $S^*$ , and  $\text{Or}_F$  is an orientation of  $F$ . The Hodge correlator map after dualisation can be viewed as a map

$$\text{Cor}_{\mathcal{H}} : \mathbb{F}_{S^*}^\bullet \longrightarrow \mathcal{D}^{*,*}(\mathcal{C}_{X,S^*}). \quad (131)$$

Recall the  $N$ -degree operator on the Hodge complex: the  $N$ -degree of a homogeneous element of the Dolbeaut bidegree  $(-s, -t)$  is  $s + t - 1$  if  $(s, t) \neq (0, 0)$ , and 0 if  $(s, t) = (0, 0)$ .

We call an edge of a decorated forest  $(F, W)$  *special* if it is an external edge decorated by a 1-form, and *regular* otherwise.

**Proposition 6.1** *One has  $\tilde{\mathbf{G}} \in \mathcal{C}^1(\mathcal{T}_{X,S^*})$ . The  $N$ -degree of  $\text{Cor}_{\mathcal{H}}(T)$  is the number of regular edges of  $T$ .*

**Proof.** Let  $\omega_{\alpha,\beta}$  be the component of  $\omega_{\alpha+\beta}$  defined by using  $\alpha$  operators  $\partial$  and  $\beta$  operators  $\bar{\partial}$ .

Let  $F$  be a plane trivalent  $S$ -decorated forest. Let  $\text{Cor}_{\mathcal{H}}(F)(\alpha, \beta)$  be the component of  $\text{Cor}_{\mathcal{H}}(F)$  defined by using the component  $\omega_{\alpha,\beta}$  of  $\omega_{\alpha+\beta}$ . Let  $(a, b)$  be its de Rham bidegree, and  $(-p, -q)$  is the Hodge bidegree. Therefore its Dolbeaut bidegree  $(-s, -t)$  is given by  $s = p - a$ ,  $t = q - b$ . The following lemma asserts that the Dolbeaut bidegree of a component of  $\text{Cor}_{\mathcal{H}}(F)$  is determined by the  $(\partial, \bar{\partial})$ -type of the component of  $\omega$  used in its definition.

**Lemma 6.2** *Let  $F$  be a plane  $S$ -decorated forest with regular edges. Then for the component  $\text{Cor}_{\mathcal{H}}(F)(\alpha, \beta)$  one has*

$$s = 1 + \beta, \quad t = 1 + \alpha.$$



**Proof.** Assume that  $F$  is decorated by forms/currents of types  $(a_i, b_i)$ . Let  $n$  be the number of non-special (i.e. not decorated by a class in  $H^1(X, \mathbb{C})$ ) edges. Then  $\alpha + \beta = n - 1$ , and

$$a = \sum_{i=1}^{m+1} a_i + \alpha - n - \pi_0(F).$$

Indeed, since the Green current is a 0-current, the sum of its first two terms here is the sum of the holomorphic degrees of the integrand. By an Euler characteristic argument, we integrate over  $n + \pi_0(F)$  copies of  $X$ . It remains to notice that  $p = \sum a_i - \pi_0(F)$ . Therefore  $p - a = n - \alpha = \beta + 1$ . The second formula is similar. The lemma is proved.

There are two decorated trees with no regular edges, see Fig 18.

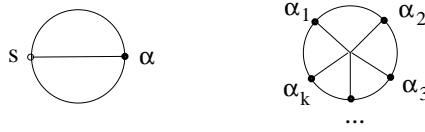


Figure 18: The decorated trees providing Hodge correlators of the Dolbeaut bidegrees  $(0, 0)$ .

If a tree  $T$  admits a decoration without regular edges, the corresponding component of the section of  $\mathcal{C}_{\mathcal{H}}^1(\mathcal{C}_{X,S})$  provided by  $T$  is defined as follows:

a)  $T$  is the left tree on Fig 18 – a single edge decorated tree with a single  $S$ -decorated vertex  $\{s\}$ . Recall the  $H_1(X/B(\mathbb{C}), \mathbb{C})$ -valued 1-form  $\nu$  on  $X \times_B X(\mathbb{C})$  defined in (60). Taking into account the base-point section  $s_0$ , there is a map  $s_0 \times s : B \rightarrow X \times_B X$ . We assign to  $T$  the element

$$\text{Cor}_{\mathcal{H}}(T) = (s_0 \times s)^* \nu \in \mathcal{C}_{\mathcal{H}}^1(\mathcal{C}_{X,S^*}).$$

b) If  $(T, W)$  is the right tree on Fig 18 decorated by 1-forms, the Hodge correlator assigned to it is a  $(k - 2)$ -form on the base  $B(\mathbb{C})$  given by the push forward of the  $k$ -form  $\alpha_1 \wedge \dots \wedge \alpha_k$  along the projection  $X(\mathbb{C}) \rightarrow B(\mathbb{C})$ . One easily sees that  $\text{Cor}_{\mathcal{H}}(T)$  lies in  $\mathcal{C}_{\mathcal{H}}^1(\mathcal{C}_{X,S})$ .

These elements are of the Dolbeaut bidegree  $(0, 0)$ . The converse is also true: Let  $T$  be an  $S^*$ -decorated tree. Then

$$\text{The Dolbeaut bidegree of } \text{Cor}_{\mathcal{H}}(T) \text{ is } (0, 0) \iff T \text{ is as on Fig 18,} \quad (132)$$

The Proposition follows immediately from this and Lemma 6.2.

Let us set

$$\omega_{\alpha,\beta}^* := \binom{\alpha + \beta}{\alpha} \omega_{\alpha,\beta}, \quad \omega_n^* := \sum_{\alpha + \beta = n} \omega_{\alpha,\beta}^* \quad (133)$$

Denote by  $\text{Cor}_{\mathcal{H}}^*$  the version of the Hodge correlator defined using the forms  $\omega_n^*$  instead of  $\omega_n$ . Just like in Proposition 6.1, after the dualization we get an element

$$\mathbf{G}^* \in \mathcal{C}_{\mathcal{H}}^1(\mathcal{C}\mathcal{L}ie_{X,S^*}).$$

The variation  $\mathcal{C}\mathcal{L}ie_{X,S^*}$  is a Lie algebra in the category of variations of real Hodge structures over  $B(\mathbb{C})$  (Section 5). Therefore the complex  $\mathcal{C}_{\mathcal{H}}^{\bullet}(\mathcal{C}\mathcal{L}ie_{X,S^*})$  with the differential  $\delta$  (Section 4) has a structure of a DG Lie algebra in the same category. The Lie bracket  $[\cdot, \cdot]$  given by the product  $*$ , followed by the commutator map in the Lie algebra  $\mathcal{C}\mathcal{L}ie_{X,S^*}$ .

The main result of this section is the following theorem.

**Theorem 6.3** *The Hodge correlator  $\mathbf{G}^*$  satisfies the Maurer-Cartan equations*

$$\delta \mathbf{G}^* + [\mathbf{G}^*, \mathbf{G}^*] = 0.$$

The Hodge correlator map for general, not necessarily trivalent plane trees provides an element

$$\tilde{\mathbf{G}}^* \in \mathcal{C}_{\mathcal{H}}^1(\mathcal{T}_{X,S^*}).$$

The variation  $\mathcal{T}_{X,S^*}^\vee$  has a structure of a DG Lie coalgebra in the category of variations of Hodge structures over  $B(\mathbb{C})$ . Therefore its dual  $\mathcal{T}_{X,S^*}$  is a DG Lie algebra in the same category. Thus the complex  $\mathcal{C}_{\mathcal{H}}^\bullet(\mathcal{T}_{X,S^*})$  is a DG Lie algebra in the same category.

**Theorem 6.4** *The Hodge correlator  $\tilde{\mathbf{G}}^*$  satisfies the Maurer-Cartan equations*

$$\delta \tilde{\mathbf{G}}^* + [\tilde{\mathbf{G}}^*, \tilde{\mathbf{G}}^*] = 0.$$

Recall the injective homomorphism from the Lie coalgebra  $\mathcal{C}_{X,S^*}^\vee$  to the DG Lie coalgebra  $\mathcal{T}_{X,S^*}^\vee$  (Section 5.2). Dualizing it, we get a surjective homomorphism from the DG Lie algebra  $\mathcal{T}_{X,S^*}$  to the Lie algebra  $\mathcal{C}_{X,S^*}$ . Therefore there is a natural projection of DG Lie algebras

$$\mathcal{C}_{\mathcal{H}}^\bullet(\mathcal{T}_{X,S^*}) \longrightarrow \mathcal{C}_{\mathcal{H}}^\bullet(\mathcal{C}_{X,S^*}).$$

By the very definition it sends  $\tilde{\mathbf{G}}^*$  to  $\mathbf{G}^*$ . Theorem 6.3 follows from this and Theorem 6.4.

We prove Theorem 6.4 in two steps.

(i) In section 6.2 we show that the Hodge correlator  $\tilde{\mathbf{G}}^*$  is multiplicative. In fact our definition of the  $*$ -product in the Hodge complexes was suggested by computation of the Hodge correlator map for a union of two trees.

(ii) We prove a formula for the differential of the Hodge correlator in Theorem 6.8. In fact our definition of the differential  $\partial$  on  $\mathcal{F}_{H,S^*}^{\vee,\bullet}$  was suggested by computation of the differential of the Hodge correlator. Combining the multiplicativity with Theorem 6.8 we prove Theorem 6.4.

## 6.2 Multiplicativity of the Hodge correlators

Lemma 6.2 implies that the Hodge correlator map  $\text{Cor}_{\mathcal{H}}^*$ , viewed as a map (131), is a map to the Hodge complex:

$$\text{Cor}_{\mathcal{H}}^* : \mathbb{F}_{S^*}^\bullet \longrightarrow \mathcal{C}_{\mathcal{H}}^\bullet(\mathcal{C}_{X,S^*}). \quad (134)$$

One easily sees that it respects the gradings.

Recall that  $\mathcal{C}_{\mathcal{H}}^\bullet(\mathcal{C}_{X,S})$  has a graded commutative algebra structure with the product  $*$ .

**Proposition 6.5** *The Hodge correlator map (134) is multiplicative:*

$$\text{Cor}_{\mathcal{H}}^*(T' \circ T'') = \text{Cor}_{\mathcal{H}}^*(T') * \text{Cor}_{\mathcal{H}}^*(T'').$$

**Proof.** Let us consider first the case when none of the trees  $T'$  and  $T''$  is as on Fig 18. We use some multiples of the forms involved whose components factorize nicely. Let

$$\tilde{\omega}_{\alpha,\beta}^* := (\alpha + \beta + 1) \omega_{\alpha,\beta}^*. \quad (135)$$

Then  $\tilde{\omega}^*$  is the unique positive multiple of  $\omega_{\alpha+\beta}$  whose  $(\alpha, \beta)$ -component has every summand with the coefficient  $\pm 1$ . In the form  $\eta_m$  every summand also has the coefficient  $\pm 1$ . This implies a factorization

$$\tilde{\omega}_{\alpha,\beta}^* = \sum_{\alpha'+\alpha''=\alpha, \beta'+\beta''=\beta} \left( \tilde{\omega}_{\alpha',\beta'}^* \times \eta_{\alpha''+\beta''+1} + \eta_{\alpha'+\beta'+1} \times \tilde{\omega}_{\alpha'',\beta''}^* \right)$$

of currents on

$$X_{/B}(\mathbb{C})\{\text{vertices of } T'\} \times X_{/B}(\mathbb{C})\{\text{vertices of } T''\}. \quad (136)$$

In particular  $\alpha' + \beta' + 1$  in the sum is the number of edges of  $T'$ , and similarly for  $T''$  and  $T' \cup T''$ .

Let  $\xi_{\alpha,\beta}$  be the  $(\alpha, \beta)$ -component of  $\xi_{\alpha+\beta}$ . There are the following identities of currents:

$$\eta_{\alpha+\beta+1} \stackrel{(54)}{=} \mathbf{d}^C \xi_{\alpha+\beta}, \quad \xi_{\alpha,\beta} = \omega_{\alpha,\beta}^*,$$

Combining them we arrive at the following crucial factorization of currents on (136):

$$\tilde{\omega}_{\alpha,\beta}^* = \sum_{\alpha'+\alpha''=\alpha, \beta'+\beta''=\beta} \left( \tilde{\omega}_{\alpha',\beta'}^* \times \mathbf{d}^C \omega_{\alpha'',\beta''}^* + \mathbf{d}^C \omega_{\alpha',\beta'}^* \times \tilde{\omega}_{\alpha'',\beta''}^* \right). \quad (137)$$

We view an element  $\text{Cor}_{\mathcal{H}}^*(F)$  as a linear functional on the decoration space  $S^\bullet \mathcal{C}_{H,S}^\vee$ . Denote by  $\text{Cor}_{\mathcal{H}}^*(F)_{s,t}$  the Dolbeaut bidegree  $(-s, -t)$  component of  $\text{Cor}_{\mathcal{H}}^*(F)$ . Lemma 6.2 asserts that  $\omega_{\alpha,\beta}^*$  delivers the component  $\text{Cor}_{\mathcal{H}}^*(F)_{\beta-1, \alpha-1}$ .

A decoration of  $T' \cup T''$  is a product of decoration of  $T'$  and  $T''$ , and the same has to be true for the orientations. Therefore multiplying (137) by the decorations, integrating, taking the sum over all plane trivalent trees with the chosen decorations, and using definition (135), we get

$$\begin{aligned} (s+t-1)\text{Cor}_{\mathcal{H}}^*(T' \circ T'')_{s,t} = \\ \sum_{s'+s''=s, t'+t''=t} \left( (s'+t'-1)\text{Cor}_{\mathcal{H}}^*(T')_{s',t'} \otimes \mathbf{d}^C \text{Cor}_{\mathcal{H}}^*(T'')_{s'',t''} \right. \\ \left. + (s''+t''-1)\mathbf{d}^C \text{Cor}_{\mathcal{H}}^*(T')_{s',t'} \otimes \text{Cor}_{\mathcal{H}}^*(T'')_{s'',t''} \right). \end{aligned}$$

Comparing this formula with formula (81) for the  $*$ -product we conclude that Proposition 6.5 is proved in the case when none of the trees  $T'$  and  $T''$  is as on Fig 18.

Now suppose that at least one of the trees  $T'$ ,  $T''$  is as on Fig 18. Assume first that just one of them, say the tree  $T''$ , is as on Fig 18. Then an argument similar to the one above shows that

$$(s+t-1)\text{Cor}_{\mathcal{H}}^*(T' \circ T'')_{s,t} = \sum_{s'+s''=s, t'+t''=t} (s'+t'-1)\text{Cor}_{\mathcal{H}}^*(T')_{s',t'} \otimes (\mathbf{1} * \text{Cor}_{\mathcal{H}}^*(T''))_{s'',t''} \quad (138)$$

where  $s'' + t'' = 1$ . Indeed, in this case the contribution of the tree  $T''$  is given by a sum  $\psi(T'') + \eta(T'')$ , where  $\psi(T'')$  is the component provided by the decorations of  $T''$  without regular edges. By Lemma 6.2 the Dolbeaut bidegree of  $\psi(T'')$  is  $(0, 0)$ , so  $\mathbf{1} * \psi(T'') = \psi(T'')$ . The section  $\eta(T'')$  is obtained by applying  $\mathbf{d}^C$  to the component of the correlator of  $T''$  provided by the decorations of  $T''$  with a regular edge. Thanks to (132), the Dolbeaut bidegree of the latter correlator is different from  $(0, 0)$ , so the operator  $\mathbf{1}*$  acts on it as  $\mathbf{d}^C$ .

Similarly, if both  $T'$  and  $T''$  are as on Fig 18, the previous argument give us

$$\text{Cor}_{\mathcal{H}}^*(T' \circ T'')_{1,1} = \sum_{s'+s''=s, t'+t''=t} (\mathbf{1} * \text{Cor}_{\mathcal{H}}^*(T'))_{s',t'} \otimes (\mathbf{1} * \text{Cor}_{\mathcal{H}}^*(T''))_{s'',t''}. \quad (139)$$

Comparing (138)-(139) with formula (81) for the  $*$ -product we see that Proposition 6.5 is proved.

### 6.3 Differential equations for the Hodge correlators

Recall the specialization from Section 2.2. One defines similarly specializations of forms by taking specializations of their coefficients.

**Lemma 6.6** *Let  $W = \mathcal{C}(\{s\} \otimes W_0)$ . Then the specialization  $\mathrm{Sp}_{v_0}^{s \rightarrow s_0} \mathrm{Cor}_{\mathcal{H}, v_0}(W) = 0$ .*

**Proof.** If  $W$  is different from  $\mathcal{C}(\{s\} \otimes \alpha)$  this follows immediately from the property (59) of the Green function assigned to the edge decorated by  $\{s\}$ . Further, the restriction of  $\mathrm{Cor}_{\mathcal{H}} \mathcal{C}(\{s\} \otimes \alpha)$  to the divisor  $s = s_0$  is zero by the very definition. The lemma is proved.

The element  $\tilde{\mathbf{G}}$  is nothing else but a map

$$\mathrm{Cor}_{\mathcal{H}} : \mathcal{T}_{X, S^*}^{\vee} \longrightarrow \mathcal{D}_{B(\mathbb{C})}^{*,*}.$$

Recall that  $\mathrm{Cor}_{\mathcal{H}}(T, W; \mathrm{Or}_T)$  is calculated as follows. Write  $W = \mathcal{C}(\alpha_0 \otimes \dots \otimes \alpha_k)$ , where  $\alpha_i$  are sections of the local system  $V_{X, S^*}^{\vee}$ . Pick forms  $\beta_i$  representing sections  $\alpha_i$ . Then the correlator is given by the push forward of the form

$$\omega_m(G_{E_0} \wedge \dots \wedge G_{E_m}) \wedge \beta_0 \otimes \dots \otimes \beta_k. \quad (140)$$

Let  $\mathbf{P} = c_1 \mathbf{d}' + c_2 \mathbf{d}'' + c_3 \mathbf{d}' \mathbf{d}''$ , and  $P = c_1 d' + c_2 d'' + c_3 d' d''$ .

**Lemma 6.7** *The value of  $\mathbf{P} \mathrm{Cor}_{\mathcal{H}}$  on  $W$  is given by the push forward of the form*

$$\left( P \omega_m(G_{E_0} \wedge \dots \wedge G_{E_m}) \right) \wedge \beta_0 \otimes \dots \otimes \beta_k. \quad (141)$$

**Proof.** The value of  $\mathbf{d}' \mathrm{Cor}_{\mathcal{H}}$  on  $(T, W)$  is given by

$$d' \left( \mathrm{Cor}_{\mathcal{H}}(T, \mathcal{C}(\beta_0 \otimes \dots \otimes \beta_k)) \right) - \pm \sum_{i=0}^k \mathrm{Cor}_{\mathcal{H}}(T, \mathcal{C}(\beta_0 \otimes \dots \otimes d\beta_i \otimes \dots \otimes \beta_k)). \quad (142)$$

To calculate this, we have to push-forward to  $B(\mathbb{C})$  a current obtained by the following procedure: apply  $d'$  to the current (140), and subtract from the obtained form an alternated sum over  $i = 0, \dots, k$  of the currents

$$\omega_m(G_{E_0} \wedge \dots \wedge G_{E_m}) \wedge \beta_0 \otimes d' \beta_i \otimes \dots \otimes \beta_k, \quad (143)$$

The current we get this way coincides with (141) for  $\mathbf{P} = \mathbf{d}'$ . The argument for  $\mathbf{d}''$  is completely similar. Iterating the argument, we get the claim for  $\mathbf{d}' \mathbf{d}''$ . The lemma is proved.

Denote by  $\tilde{\mathbf{G}}_{>1}$  (respectively  $\tilde{\mathbf{G}}_1$  or  $\tilde{\mathbf{G}}_0$ ) the component of  $\tilde{\mathbf{G}}$  of the  $N$ -degree  $> 1$  (respectively 1 or 0). By Proposition 6.1 it is the contribution of the decorated trees with more then one regular edge (respectively one or no regular edges). Let us consider the composition

$$\mathcal{T}_{X, S^*}^{\vee} \xrightarrow{\partial} \mathcal{F}_{X, S^*}^{\vee} \xrightarrow{\mathrm{Cor}_{\mathcal{H}}} \mathcal{D}_{B(\mathbb{C})}^{*,*}. \quad (144)$$

Denote by  $\mathrm{Cor}_{\mathcal{H}} \circ \partial_{>1}$  (respectively  $\mathrm{Cor}_{\mathcal{H}} \circ \partial_1$ ) the restriction of this map to the subspace spanned by the decorated trees with more then one (respectively one) regular edge. Notice that  $\mathbf{d} \tilde{\mathbf{G}}_{>1}$  and  $\mathbf{d} \tilde{\mathbf{G}}_1$  can be viewed as similar maps.

**Theorem 6.8** *One has*

$$\mathbf{d}\tilde{\mathbf{G}}_{>1} = \text{Cor}_{\mathcal{H}} \circ \partial_{>1}, \quad \mathbf{d}\mathbf{d}^{\mathbf{C}}\tilde{\mathbf{G}}_1 = \text{Cor}_{\mathcal{H}} \circ \partial_1, \quad \mathbf{d}\tilde{\mathbf{G}}_0 = 0. \quad (145)$$

**Proof.** We start from the left identity in (145). Thanks to Lemma 6.7 the calculation of the value of  $\mathbf{d}\tilde{\mathbf{G}}_{>1}$  on a decorated tree  $(T, W)$  boils down to the calculation of  $d\omega_m(G_{E_0} \wedge \dots \wedge G_{E_m})$  – the differentials of the decorations do not appear in the final result.

We employ formula (48) for  $d\omega_m(\varphi_0, \dots, \varphi_m)$ , and use formula (56) for calculation of the Laplacian  $\Delta G_{s_0}(x, y)$ , where  $\Delta = \mathbf{d}\mathbf{d}^{\mathbf{C}}$ , which appears in this formula. There are three terms in the formula for the Laplacian:

- (a) The  $\delta_{\Delta_X}$ -term.
- (b) The volume term  $p_1^* \delta_{s_0} + p_2^* \delta_{s_0}$ .
- (c) The reduced Casimir term – the remaining term.

We call the sum of the terms (b)+(c) the Casimir term.

Let us calculate the contribution of the Laplacian corresponding to a regular edge  $E$ . We will show that each term of the contribution matches certain term in the form  $\text{Cor}_{\mathcal{H}} \circ \partial$ .

Consider the following cases:

1.  $E$  is an internal edge. There are the following cases, matching the terms in  $\Delta G$ :

(1a) The  $\delta_{\Delta_X}$ -term contributes the diagram with one vertex obtained by shrinking the edge  $E$ , see the right of Fig 8. This matches the graph complex differential  $\partial_{\Delta}$ .

(1b) We claim that the contribution of the volume term in  $\Delta G_E$  is zero. Indeed, let  $T_1$  and  $T_2$  be the two trees obtained by cutting  $T$  along the edge  $E$ . We assign the current  $\delta_{s_0}$  to the end of one of them, say  $T_1$ , obtained by cutting  $E$ . Denote by  $W_1$  the induced decoration of  $T_1$ . Then, arguing just as in Section 6.2, the contribution of the tree  $T_1$  is proportional to either  $\text{Cor}_{\mathcal{H}, s_0}(W_1)$ , or  $\mathbf{d}^{\mathbf{C}}\text{Cor}_{\mathcal{H}, s_0}(W_1)$ . Let us show that in any case  $\text{Cor}_{\mathcal{H}, s_0}(W_1) = 0$ . The decoration  $W_1 = \{a\} \otimes W'_1$  delivers *a priori* divergent correlator  $\text{Cor}_{\mathcal{H}, a}(\{a\} \otimes W'_1)$ . Since the sum of all correlators involved is convergent, we can understand it via the specialization of a correlator  $\text{Cor}_{\mathcal{H}, s_0}(\{a\} \otimes W'_1)$  when  $a \rightarrow s_0$  using the tangent vector  $v_0$  at  $s_0$ . Thanks to Lemma 6.6 this specialization is zero. This settles the (1b) case.

(1c) The contribution of the reduced Casimir term matches the form assigned by the map  $\text{Cor}_{\mathcal{H}}$  to the minus of the sum of forests on Fig 9. The latter sum is the contribution of the edge  $E$  to the map  $\partial_{\text{Cas}}$ .

2.  $E$  is an external edge decorated by a section  $s$ . There are the following cases:

(2a) The contribution of the  $\delta_{\Delta_X}$ -term in  $\Delta G_E$  for a Feynman diagram  $T$  is calculated as follows. Let  $s$  be the decoration at the external vertex  $v_E$  of  $E$ . Let us cut out a small neighborhood of  $E$  from  $T$ , getting several connected Feynman diagrams,  $T_1, \dots, T_k$ . Each of them has an external vertex provided by the removed edge  $E$  (referred to as a new vertex), and the other external vertices are the ones of  $T$  minus  $v_E$ . The tree  $T_i$  is decorated: the new vertex is decorated by  $s$ , and the others inherit their decorations from  $T$ , see Fig 11. This matches the component  $\partial_S$  of the differential  $\partial$ .

(2b) The contribution of the volume term in  $\Delta G_E$  is zero. It is similar to (1b), except that one needs add an argument for the contribution of  $\delta_{s_0}$  related to an external vertex  $v$  of  $E$ . Let  $s$  be the decoration at  $v$ . Then the contribution is  $\delta(s_0 - s)$ . It does not affect the result since the divisor  $S$  is smooth over  $B$ , so when we move over  $B$  the point  $s$  can not collide with  $s_0$ .

(2c). It is similar to (1c). The contribution of the reduced Casimir term in  $\Delta G_E$  is negative of the form on  $X(\mathbb{C})^{\{S\text{-vertices}\}}$  given by the following recepee, illustrated on Fig 10. Cutting the edge  $E$ , we get a connected Feynman diagram  $T'$ , and a two vertex graph. The latter is decorated by a form  $\alpha_k^\vee$  and  $s$ . The form  $\alpha_k^\vee$  is viewed as a form on the  $s$ -copy of  $X(\mathbb{C})$ . The tree  $T'$  is decorated: the inherited from  $T$  external vertices inherit their decoration, and the new external vertex is decorated by  $\alpha_k$ . It gives rise to a Hodge correlator, which is a form on

$$X(\mathbb{C})^{\{S\text{-decorated vertices of } T\} - \{s\}}.$$

Taking the sum over  $k$  of the product of this correlator and the form above we get the desired form.

Let us consider now the middle identity in (145). In this case there is a single edge  $E$  to which we assign the Green function  $\omega_m(G_{E_0}) = G_E$ , wee Fig 19. Since  $dd^c$  is a Laplacian, we

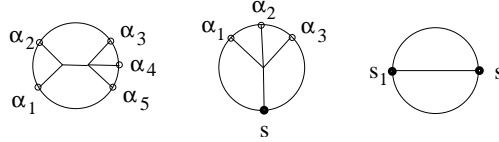


Figure 19: The decorated trees providing Hodge correlators of the Dolbeaut bidegree  $(-1, -1)$ .

proceed just like in the previous case. The right identity in (145) is trivial. Theorem 6.8 is proved.

The element  $\mathbf{G}$  is nothing else but a map  $\text{Cor}_{\mathcal{H}} : \mathcal{C}_{X,S^*}^\vee \longrightarrow \mathcal{D}_{B(\mathbb{C})}^{*,*}$ . Just as above, we decompose it into the components:  $\mathbf{G} = \mathbf{G}_{>1} + \mathbf{G}_1 + \mathbf{G}_0$ . Recall the embeddings  $\Lambda^k \mathcal{C}_{X,S^*}^\vee \hookrightarrow \mathcal{F}_{X,S^*}^{\vee,k}$  (Section 6.2). By Lemma 5.3 the composition (144) restricts to the composition  $\mathcal{C}_{X,S^*}^\vee \xrightarrow{\delta} \Lambda^2 \mathcal{C}_{X,S^*}^\vee \xrightarrow{\text{Cor}_{\mathcal{H}}} \mathcal{D}_{B(\mathbb{C})}^{*,*}$ . We decompose the cobracket  $\delta = \delta_{>1} + \delta_1$ . Corollary 6.9 below follows immediately from Theorem 6.8.

**Corollary 6.9** *One has  $d\mathbf{G}_{>1} = \text{Cor}_{\mathcal{H}} \circ \delta_{>1}$ ,  $dd^c \mathbf{G}_1 = \text{Cor}_{\mathcal{H}} \circ \delta_1$ , and  $d\mathbf{G}_0 = 0$ .*

**Proof of Theorem 6.4.** Using the modified map  $\text{Cor}_{\mathcal{H}}^*$  in Corollary 6.9, we have<sup>6</sup>

$$\text{Cor}_{\mathcal{H}}^* \circ \delta_{\text{Colie}} = -\frac{1}{2} \delta_{\text{Hod}} \mathbf{G}^*.$$

Here on the left stands the composition

$$\mathcal{C}_{X,S^*}^\vee \xrightarrow{\delta_{\text{Colie}}} \Lambda^2 \mathcal{C}_{X,S^*}^\vee \xrightarrow{\text{Cor}_{\mathcal{H}}^*} \mathcal{D}_{B(\mathbb{C})}^{*,*}. \quad (146)$$

Indeed, using the operator  $\mu$ , see (82), the maps  $\text{Cor}_{\mathcal{H}}$  and  $\text{Cor}_{\mathcal{H}}^*$  are related as follows:

$$\text{Cor}_{\mathcal{H}} = \mu^{-1} \text{Cor}_{\mathcal{H}}^*.$$

This follows from formula (133) relating  $\omega_{\alpha,\beta}$  and  $\omega_{\alpha,\beta}^*$ , and Lemma 6.2. And  $2D = -\mu^{-1} \delta \mu$  by Definition 3.6.

<sup>6</sup>Do not mix the Lie coalgebra differential  $\delta = \delta_{\text{Colie}}$  with the Hodge complex differential  $\delta = \delta_{\text{Hod}}$ .

Dualizing, we view composition (146) as an element of  $\mathcal{C}_{\mathcal{H}}^2(\mathcal{C}_{X,S})$ . We claim that it equals  $\frac{1}{2}[\mathbf{G}^*, \mathbf{G}^*]$ . Indeed, by Proposition 6.5 the dual of map  $\text{Cor}_{\mathcal{H}}^*$  in (146) is the  $*$ -product  $\mathbf{G}^* * \mathbf{G}^*$ , and the dual of  $\delta_{\text{Colie}}$  is the commutator in the Lie algebra  $\mathcal{C}_{X,S}$ . Theorem 6.3 is proved.

## 7 The Lie algebra of special derivations

The pronilpotent completion  $\pi_1^{\text{nil}}(X - S, v_0)$  was described in Section 1.2. It is a pro-Lie algebra over  $\mathbb{Q}$ . It carries a weight filtration. The corresponding associate graded  $L_{X,S}$  has a simple description in terms of the cohomology of  $X$  and  $S$ . We describe the Lie algebra of all special derivations of  $L_{X,S}$ , and identify it with the Lie algebra  $\mathcal{C}_{X,S}$  from Section 5.

### 7.1 The linear algebra set-up

Let  $H$  be a finite dimensional vector space with a symplectic structure  $\omega^\vee \in \Lambda^2 H^\vee$ . Let  $\{p_i, q_i\}$  be a symplectic basis in  $H$ :  $\omega^\vee(p_i, q_i) = 1, \omega^\vee(p_i, p_j) = \omega^\vee(q_i, q_j) = 0$ . There is the dual of  $\omega^\vee$ :

$$w = \sum_i p_i \wedge q_i \in \Lambda^2 H.$$

Let  $S$  be a non-empty finite set. We need the following associative/Lie algebras:

1.  $A_{H,S}$ : the tensor algebra of  $H \oplus \mathbb{Q}[S]$ . Then  $\omega$  provides an element  $[p, q] := \sum_i [p_i, q_i] \in A_{H,S}$ . Denote by  $X_s$  the generator corresponding to  $s \in S$ .
2.  $L_{H,S}$ : the free Lie algebra generated by  $H \oplus \mathbb{Q}[S]$ . The algebra  $A_{H,S}$  is identified with its universal enveloping algebra.
3.  $\overline{A}_{H,S}$ : the quotient of the algebra  $A_{H,S}$  by the two-sided ideal generated by the element

$$[p, q] + \sum_{s \in S} X_s. \quad (147)$$

4.  $\overline{L}_{H,S}$ : the quotient of  $L_{H,S}$  by the ideal generated by (147).

When  $S$  is empty, set  $A_H := A_{H,\emptyset}, L_H := L_{H,\emptyset}$ , etc. Let  $0$  be an element of  $S$ . Set  $S^* := S - \{0\}$ . Then there are canonical isomorphisms

$$L_{H,S^*} \xrightarrow{\sim} \overline{L}_{H,S}, \quad A_{H,S^*} \xrightarrow{\sim} \overline{A}_{H,S}. \quad (148)$$

Indeed, one can express  $X_0$  via the other generators:

$$X_0 = - \sum_{s \in S^*} X_s - [p, q]. \quad (149)$$

**Example.** Let  $X$  be a smooth complex compact curve,  $H := H_1(X; \mathbb{Q})$ . The symplectic form  $\omega^\vee$  is the intersection form on  $H_1(X; \mathbb{Q})$ . Let  $S$  be a non-empty subset of points of  $X$ . Then  $\overline{L}_{H,S}$  is isomorphic, non-canonically, to  $\pi_1^{\text{nil}}(X - S, v_0)$ . It is canonically identified with the associate graded for the weight filtration  $L_{X,S}$  of the latter.

## 7.2 Special derivations of the algebra $\overline{A}_{H,S}$

The constructions of this subsection generalize the ones of Drinfeld [Dr] and Kontsevich [K]. A derivation  $\mathcal{D}$  of the algebra  $A_{H,S^*}$  is *special* if there exist elements  $B_s, s \in S^*$  such that

$$\mathcal{D}(X_0) = 0, \quad \mathcal{D}(X_s) = [B_s, X_s], \quad s \in S^*.$$

Special derivations form a Lie algebra denoted  $\text{Der}^S A_{H,S^*}$ . We define special derivations of the algebra  $\overline{A}_{H,S}$  via the isomorphism (148).

**Remark.** If  $H \neq 0$ , a collection of elements  $\{B_s\}$  does not determine a special derivation. For instance if  $S^*$  is empty this collection is also empty while, as we show below, the space of special derivations is not. If  $H = 0$  it does, see Section 4 in [G4].

Let  $A$  be an associative algebra. Recall the projection  $\mathcal{C} : A \mapsto \mathcal{C}(A) = A/[A, A]$ .

If  $B$  is a free associative algebra generated by a finite set  $\mathcal{S}$ , then there are linear maps

$$\partial/\partial X_s : \mathcal{C}(B) \longrightarrow B; \quad \mathcal{C}(X_{s_1} X_{s_2} \dots X_{s_k}) \longmapsto \sum_{s_i=s} X_{s_{i+1}} X_{s_{i+2}} \dots X_{s_{i-1}}.$$

These maps are components of the non-commutative differential map

$$D : \mathcal{C}(B) \longrightarrow B \otimes \mathbb{Q}[S], \quad D(F) = \sum_{s \in S} \frac{\partial F}{\partial X_s} \otimes X_s. \quad (150)$$

According to formula (96) in Section 4 of [G4] (an easy exercise) one has

$$\sum_{s \in S} \left[ \frac{\partial F}{\partial X_s}, X_s \right] = 0. \quad (151)$$

In particular, there are linear maps  $\partial/\partial p_i, \partial/\partial q_i, \partial/\partial X_s : \mathcal{C}(A_{H,S^*}) \rightarrow A_{H,S^*}, s \in S^*$ .

**Remark.** We can not define these maps for  $s \in S$  acting on  $\overline{A}_{H,S}$  since relation (147) will not be killed.

The vector space  $\mathcal{C}(A_{H,S^*})$  is decomposed into a direct sum of  $\mathbb{Q} \cdot 1$  and of  $\mathcal{C}^+(A_{H,S^*})$ .

**Lemma 7.1** *There is a map*

$$\kappa : \mathcal{C}^+(A_{H,S^*}) \longrightarrow \text{Der}^S(A_{H,S^*}).$$

*such that given an element  $F \in \mathcal{C}^+(A_{H,S^*})$ , the derivation  $\kappa_F$  acts on the generators by*

$$X_0 \longmapsto 0; \quad p_i \longmapsto -\frac{\partial F}{\partial q_i}; \quad q_i \longmapsto \frac{\partial F}{\partial p_i}; \quad X_s \longmapsto [X_s, \frac{\partial F}{\partial X_s}], \quad s \in S^*. \quad (152)$$

**Proof.** It follows from (151) that  $[\frac{\partial F}{\partial q_i}, q_i] + [\frac{\partial F}{\partial p_i}, p_i] + \sum_{s \in S^*} [\frac{\partial F}{\partial X_s}, X_s] = 0$ . This is equivalent to  $\kappa_F([p, q] + \sum_{s \in S^*} X_s) = 0$ , which just means that  $\kappa_F(X_0) = 0$ , and thus  $\kappa_F$  is a special derivation. The lemma is proved.

The space  $\mathcal{C}(A_{H,S^*})$  is identified, as a vector space, with the Lie algebra  $\mathcal{C}_{H,S^*}$  from Section 6. So it inherits a Lie algebra structure.



**Proposition 7.2** a) The Lie bracket on  $\mathcal{C}(A_{H,S^*})$  is given by the following formula:

$$\{F, G\} := \mathcal{C} \left( 2 \sum_{s \in S^*} \left[ \frac{\partial F}{\partial X_s}, \frac{\partial G}{\partial X_s} \right] \cdot X_s + \sum_i \left( \frac{\partial F}{\partial p_i} \cdot \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \cdot \frac{\partial G}{\partial p_i} \right) \right). \quad (153)$$

b) The map  $\kappa$  is a Lie algebra morphism.

**Proof.** a) Follows immediately from the definitions. The first term in formula (153) corresponds to the map  $\delta_S$ , and the second  $\delta_{\text{Cas}}$ . In particular this shows that the bracket (153) satisfies the Jacobi identity.

b) We will check it in the more general motivic set-up in Section 9.

Let  $\mathcal{C}'(A_{H,S^*})$  be the quotient of the vector space  $\mathcal{C}(A_{H,S^*})$  by  $\oplus_{s \in S^*} \mathbb{Q}[X_s]$ . The Lie bracket (153) descends to a Lie bracket on  $\mathcal{C}'(A_{H,S^*})$ ,

**Proposition 7.3** The map  $\kappa' : \mathcal{C}'(A_{H,S^*}) \longrightarrow \text{Der}^S A_{H,S^*}$  is a Lie algebra isomorphism.

**Proof.** Since the algebra  $A_{H,S^*}$  is free, the centralizer of  $X_s$  is  $\mathbb{Q}[X_s]$ . So  $\text{Ker} \kappa' = \oplus_s \mathbb{Q}[X_s]$ . We get an injective Lie algebra map  $\kappa' : \mathcal{C}'(A_{H,S^*}) \hookrightarrow \text{Der}^S A_{H,S^*}$ . So it remains to check that  $\kappa'$  is surjective. We reduce this to the known case when  $\dim H = 0$ . Observe that a system  $B = (B_{q_i}, B_{p_i}, B_s)$  of elements of  $A_{S^*}$ , where  $i = 1, \dots, \dim H$  and  $s \in S$ , satisfying

$$\sum_{i=1}^{\dim H} ([B_{q_i}, q_i] + [B_{p_i}, p_i]) + \sum_{s \in S} [B_s, X_s] = 0 \quad (154)$$

defines a special derivation  $D_B$  of the algebra  $A_{S^*}$ , acting on the generators as follows:

$$D_B : p_i \longmapsto -B_{q_i}, \quad q_i \longmapsto B_{p_i}, \quad X_s \longmapsto B_s.$$

Indeed,  $D_B([p, q] + \sum_{s \in S} X_s)$  is given by the left hand side of (154). So by Proposition 5.1 of [G4], there exists an  $F \in \mathcal{C}(A_{S^*})$  such that  $B_{q_i} = -\partial F / \partial q_i$ ,  $B_{p_i} = \partial F / \partial p_i$ , and  $X_s = \partial F / \partial X_s$ . The proposition is proved.

**Proposition 7.4** The map  $\kappa : \mathcal{CLie}_{H,S^*} \longrightarrow \text{Der}^S L_{H,S^*}$  is a Lie algebra isomorphism.

**Proof.** Let us consider a Casimir element

$$\xi := \sum_W \frac{1}{|\text{Aut}(W)|} W^\vee \otimes W$$

where the sum is over a basis  $\{W\}$  in  $\mathcal{C}_{H,S^*}$ , and  $\{W^\vee\}$  is the dual basis. Here  $\text{Aut}(W)$  is the automorphism group of the cyclic word  $W$ . There is a coproduct  $\Delta$  in the algebra  $A_{H,S^*}$  determined by the property that the generators of the algebra are primitive. It dualizes the shuffle product  $\circ_{\text{Sh}}$ . Let  $\overline{\Delta}(Z) := \Delta(Z) - (1 \otimes Z + Z \otimes 1)$  be the reduced coproduct. Then  $L_{H,S^*} = \text{Ker} \overline{\Delta}$ . Choose a basis  $\{Y_i\}$  in  $V_{H,S^*}$ . Given a basis vector  $Y$ , consider the expression

$$\text{id} \otimes \overline{\Delta} \circ \mathcal{D}_Y(\xi) \in \mathcal{C}_{H,S^*}^\vee \otimes A_{H,S^*}. \quad (155)$$

Choose two elements  $A, B$  of the induced basis in  $A_{H,S^*}$ . Then  $A \otimes B \in A_{H,S^*} \otimes A_{H,S^*}$  appears in (155) with coefficient  $\mathcal{C}(Y(A \circ_{\text{Sh}} B))$ . Notice that the factor  $1/|\text{Aut}(W)|$  is necessary. The proposition is proved.

### 7.3 A symmetric description of the Lie algebra of special derivations

To define a special derivation we have to choose a specific element, 0, of the set  $S$ . Let us show that the Lie algebra of all special derivations can be defined using only the symplectic vector space  $H$  and the set  $S$ . However the way it acts by special derivations of  $A_{H,S^*}$  depends on the choice of  $0 \in S$ .

Let us define a Lie algebra structure on the vector space  $\mathcal{C}(A_{H,S})$  via formula (153) where  $S^*$  is replaced by  $S$ . Then we prove the following result.

**Proposition 7.5** *Take the two-sided ideal in the algebra  $A_{H,S}$ , generated by  $[p, q] + \sum_{s \in S} X_s$ . Its image in the Lie algebra  $\mathcal{C}(A_{H,S})$  is an ideal. So the quotient  $\mathcal{C}(\overline{A}_{H,S})$  of  $\mathcal{C}(A_{H,S})$  by this ideal is a Lie algebra. The canonical map  $i_0 : \mathcal{C}(\overline{A}_{H,S}) \rightarrow \mathcal{C}(A_{H,S^*})$  is a Lie algebra isomorphism.*

**Proof.** It is an easy exercise left to the reader.<sup>7</sup>

So for any element  $s \in S$  there is a Lie algebra isomorphism

$$\kappa_s : \mathcal{C}(\overline{A}_{H,S}) \xrightarrow{\sim} \text{Der}^S A_{H,S-s}. \quad (156)$$

The isomorphism  $i_s : A_{H,S-\{s\}} \rightarrow \overline{A}_{H,S}$  provides a Lie algebra isomorphism  $\mathcal{C}(A_{H,S-\{s\}}) \rightarrow \mathcal{C}(\overline{A}_{H,S})$ . Furthermore, for any  $s \in S$  isomorphism (156) restricts to a Lie algebra isomorphism

$$\kappa_s : \mathcal{CLie}(\overline{L}_{H,S}) \xrightarrow{\sim} \text{Der}^S L_{H,S-s}. \quad (157)$$

There is a Lie algebra isomorphism  $\mathcal{CLie}(L_{H,S-\{s\}}) \rightarrow \mathcal{C}(\overline{L}_{H,S})$ .

**The Lie algebra of outer semi-special derivations.** We say that a derivation  $\mathcal{D}$  of the associative algebra  $\overline{A}_{H,S}$  is *semi-special* if it preserves the conjugacy classes of each of the generators  $X_s$ ,  $s \in S$ , i.e. if there exist elements  $B_s$  of the algebra such that

$$\mathcal{D}(X_s) = [B_s, X_s], \quad s \in S.$$

The Lie algebra of semi-special derivations is denoted by  $\text{Der}^{SS} \overline{A}_{H,S}$ . An inner derivation is a particular example of a semi-special derivation. We define the Lie algebra of *outer semi-special* derivations of the algebra  $\overline{A}_{H,S}$  by taking the quotient modulo the Lie subalgebra of inner derivations:

$$\text{ODer}^{SS} \overline{A}_{H,S} := \frac{\text{Der}^{SS} \overline{A}_{H,S}}{\text{InDer}^{SS} \overline{A}_{H,S}}.$$

For an  $s \in S$ , let  $\text{Der}_s^S \overline{A}_{H,S} \subset \text{Der}^{SS} \overline{A}_{H,S}$  be the Lie subalgebra of all derivations special for the generator  $X_s$ . It consists of all semi-special derivations killing  $X_s$ . Thus there is a Lie algebra map

$$p_s : \text{Der}_s^S \overline{A}_{H,S} \rightarrow \text{Der}^{SS} \overline{A}_{H,S}.$$

**Lemma 7.6** *The map  $p_s$  is surjective. Its kernel is spanned by the inner derivation  $[X_s^n, *]$ .*

**Proof.** Let  $\mathcal{D}$  be a semi-special derivation, such that  $\mathcal{D}(X_s) = [B_s, X_s]$ . Subtracting the inner derivation  $[B_s, *]$  from  $\mathcal{D}$  we get a derivation special with respect to the generator  $X_s$ . So  $p_s$  is surjective. Since  $\overline{A}_{H,S-\{s\}}$  is a free associative algebra, we get the second claim.

<sup>7</sup>A natural proof of is given by deducing it from a general result in the DG version of the story.

## 8 Variations of mixed $\mathbb{R}$ -Hodge structures via Hodge correlators

**A variation of  $\mathbb{R}$ -MHS on  $\pi_1^{\text{nil}}(X-S, v_0)$  via Hodge correlators.** Recall that  $S = S^* \cup \{s_0\}$ . Recall (Section 1.7) the enhanced moduli space  $\mathcal{M}'_{g,n}$  parametrising collections  $S$  of  $n$  distinct points on a genus  $g$  complex curve  $X$  plus a tangent vector at  $s_0$ . There is a local system  $\mathcal{L}$  over  $\mathcal{M}'_{g,n}$  with the fiber

$$L_{X,S^*;v_0} \cong \text{gr}^W \pi_1^{\text{nil}}(X-S, v_0). \quad (158)$$

It is equipped with the Gauss-Manin connection.

A tangent vector  $v_0$  at the point  $s_0 \in X$  provides the normalized Green function  $G_{v_0}(x, y)$  (Section 2.2). It gives rise to the Hodge correlator map, and hence to an element

$$\tilde{\mathbf{G}}_{v_0} \in \mathcal{C}_{\mathcal{H}}^1(\mathcal{C}\mathcal{L}ie_{X,S}).$$

The Lie algebra  $\mathcal{C}_{X,S}$  acts by special derivations of the Lie algebra  $L_{X,S^*;v_0}$  (Section 7). So we arrive at an endomorphism  $\mathbf{G}_{v_0}$  of the smooth bundle  $\mathcal{L}_{\infty}$ . We proved in Section 6 that it satisfies the Maurer-Cartan differential equation  $\delta \mathbf{G}_{v_0} + [\mathbf{G}_{v_0}, \mathbf{G}_{v_0}] = 0$ . Therefore the construction of Section 4 provides a variation of mixed  $\mathbb{R}$ -Hodge structures. Theorem 1.9 claims that it is isomorphic to the standard variation of mixed  $\mathbb{R}$ -Hodge structures  $\pi_1^{\text{nil}}(X-S, v_0)$ .

**A simple proof of Theorem 1.9 for rational curves.** Denote by  $G_{p,p}^{\text{Cor}}$  the Green operators for the Hodge correlators on  $\mathbb{P}^1 - S$ . The standard variation of mixed  $\mathbb{R}$ -Hodge structures  $\pi_1^{\mathcal{H}}(\mathbb{P}^1 - S, v_0)$  is described by the Green operators denoted by  $G_{p,p}^{\text{st}}$ . Let us prove by induction on  $p$  that  $G_{p,p}^{\text{Cor}} = G_{p,p}^{\text{st}}$ . The base of the induction is given by the following straightforward Lemma:

**Lemma 8.1** *One has  $G_{1,1}^{\text{Cor}} = G_{1,1}^{\text{st}}$ .*

Suppose that the claim is proved for  $p < n$ . The systems of differential equations for Green operators describing variations of  $\mathbb{R}$ -MHS implies that the endomorphisms  $G_{n,n}^{\text{Cor}}$  and  $G_{n,n}^{\text{st}}$  satisfy the same differential equations, and that  $G_{n,n}^{\text{Cor}} - G_{n,n}^{\text{st}}$  is annihilated by both  $\partial$  and  $\bar{\partial}$ , and thus is a constant. To check that this constant is zero we use the shuffle relations from Proposition 2.7, which imply that a non-zero multiple of this constant is zero. This proves the theorem.

This argument does not work in the non-Tate case since we do not impose  $\partial$ -differential equations on  $G_{1,q}$ , as well as  $\bar{\partial}$ -differential equations on  $G_{p,1}$ . Let us address now the general case of Theorem 1.9.

**$\mathbb{R}$ -MHS on  $\pi_1^{\text{nil}}(X-S, a)$  with a regular base point  $a$  via Hodge correlators.** Let  $a \in X-S$ . Choose a non-zero tangent vector  $v_a$  at  $a$ . There is a canonical isomorphism

$$\pi_1^{\text{nil}}(X-S, a) = \pi_1^{\text{nil}}(X-S \cup \{a\}, v_a)/(v_a)$$

where  $(v_a)$  is the ideal generated by  $v_a$ . The Green operator on  $\text{gr}^W \pi_1^{\text{nil}}(X-S \cup \{a\}, v_a)$  is a special derivation, and thus preserves the ideal  $\text{gr}^W(v_a)$ . So it induces an operator  $\mathbf{G}_{v_a}$  on the quotient, which gives rise to an  $\mathbb{R}$ -MHS on  $\pi_1^{\text{nil}}(X-S, a)$ .

**Good unipotent variations.** R. Hain and S. Zucker [HZ] introduced a notion of a *good unipotent variation of mixed Hodge structures* on an open algebraic variety  $U$ , and proved that the category of such variations is equivalent to the category of representations  $\pi_1^{\text{nil}}(U, u) \otimes V \rightarrow V$  which are morphism of mixed Hodge structures. Equivalently, they are representations in the category mixed Hodge structures of the universal enveloping algebra  $A^{\mathcal{H}}(U, u)$  of  $\pi_1^{\text{nil}}(U, u)$  with the induced mixed Hodge structures.

Recall ([H1], Section 7) that a unipotent variation of mixed Hodge structures  $\mathcal{V}$  over a smooth complex curve  $X = \overline{X} - D$ , where  $\overline{X}$  is a smooth compactification, and  $D$  a finite number of points, is a good unipotent one if the following conditions at infinity hold:

- (i) Let  $\overline{\mathcal{V}} \rightarrow \overline{X}$  be Deligne's canonical extension of the local system  $\mathcal{V}$ . The Hodge bundles  $F^p \mathcal{V}$  are required to extend to holomorphic subbundles  $F^p \overline{\mathcal{V}}$  of  $\overline{\mathcal{V}}$ .
- (ii) Let  $N_P := \frac{1}{2\pi i} \log T_P$  be the logarithm of the local monodromy  $T_P$  around  $P \in D$ . Then

$$N_P(W_l V_x) \subset W_{l-2} V_x.$$

The second condition is so simple because we assumed that the global monodromy is unipotent.

**Theorem 8.2** *The variation  $\mathcal{L}$  of  $\mathbb{R}$ -MHS on  $\pi_1^{\text{nil}}(X - S, a)$  defined by the operator  $\mathbf{G}_{v_a}$  acting on (158), is a good unipotent variation of mixed Hodge structures over  $a \in X - S$ .*

We introduce a DGA of forms with logarithmic singularities, and use it to prove Theorem 8.2. Then we show how Theorem 8.2 implies Theorem 1.9.

**Differential forms on  $X$  with tame logarithmic singularities.** Let  $D$  be a normal crossing divisor in a smooth complex projective variety  $\overline{X}$ , and  $X := \overline{X} - D$ . Let us define a DG subalgebra  $\mathcal{A}_{\log}(X)$  of differential forms on  $X$  with *tame logarithmic singularities* on  $X$ . It is a slight modification of the space used in 1.3.10 in [L]. It is a DG subalgebra of the de Rham DG algebra of smooth forms on  $X$ . Choose local equations  $z_i$  of the irreducible components  $D_i$  of the divisor  $D$ . The space  $\mathcal{A}_{\log}(X)$  consists of forms  $\omega$  which can be represented as polynomials in  $\log |z_i|$ ,  $\partial \log |z_j|$ , and  $\overline{\partial} \log |z_k|$ ,

$$\omega = \sum_{a_i, \varepsilon'_j, \varepsilon''_k} \omega_{a_i, \varepsilon'_j, \varepsilon''_k} \log^{a_i} |z_i| \wedge \bigwedge_j (\partial \log |z_j|)^{\varepsilon'_j} \wedge \bigwedge_k (\overline{\partial} \log |z_k|)^{\varepsilon''_k},$$

whose coefficients are smooth functions on  $\overline{X}$  with the following property: a coefficient of a monomial containing (in positive degree) any of the three expressions  $\log |z_i|$ ,  $\partial \log |z_i|$ , and  $\overline{\partial} \log |z_i|$  related to the stratum  $D_i$  vanishes at the stratum  $D_i$ . For example,  $z_1 \log |z_1|$  belongs  $\mathcal{A}_{\log}(X)$ , while  $z_1 \log |z_2|$  does not. The space  $\mathcal{A}_{\log}(X)$  clearly does not depend on the choice of local equations  $z_i$ . The space  $\mathcal{A}_{\log}(X)$  is closed under  $\partial$  and  $\overline{\partial}$ .

**Proposition 8.3** *Choose a local parameter  $t$  at  $s$  such that  $dt$  is dual to the tangent vector  $v_s$  at  $s$ . Then  $\mathbf{G}_{v_a} - \log t \cdot \text{ad}(X_s)$  is a smooth function with tame logarithmic singularities near the point  $s$ .*

**Corollary 8.4** *The logarithm of the monodromy of the local system  $\mathcal{L}$  around  $s$  is  $\text{ad}(X_s)$ .*

**Proof.** Indeed,  $d^{\mathbb{C}} \log |t| = id \arg(t)$ , and  $d^{\mathbb{C}}$  of the other components of  $\mathbf{G}_{v_a}$  vanish at  $t = 0$  since these components have tame logarithmic singularities at  $t = 0$ . Lemma 8.4 is proved.

We prove Proposition 8.3 later on. Let us show first how it implies Theorems 8.2 and 1.9.

**Proof of Theorem 8.2.** The associate graded  $\mathrm{gr}^W \mathcal{L}$  is a constant variation. So the local system  $\mathcal{L}$  is unipotent. Thus we must check only conditions (i) and (ii).

Condition (ii) follows immediately from Corollary 8.4.

Condition (i) is just a condition on the Hodge filtration on the local system on  $X$  – it does not involve the weight filtration. So, using flatness of the twistor connection, it is sufficient to prove it for the twistor connection  $\nabla_{\mathcal{G}}^{(1)}$  restricted to  $u = 1$  instead of  $u = 0$ , see (114). The Hodge filtration for the connection  $\nabla_{\mathcal{G}}^{(1)}$  is constant. This implies the claim. Theorem 8.2 is proved.

**Proof of Theorem 1.9.** By Theorem 8.2 there is a good unipotent variation of mixed  $\mathbb{R}$ -Hodge structures on  $X - S$  given by the universal enveloping algebras  $A(X - S, a)$  of (158), when  $a$  varies. By the Hain-Zucker theorem [HZ], it is described by a morphism of  $\mathbb{R}$ -MHS

$$\theta : A^{\mathcal{H}}(X - S, a) \otimes A(X - S, a) \longrightarrow A(X - S, a).$$

Observe that  $A(X - S, a)$  is a free rank one  $A^{\mathcal{H}}(X - S, a)$ -module with a canonical generator given by the unit  $e \in A(X - S, a)$ .

**Lemma 8.5** *The action  $\theta$  coincides with the standard one.*

**Proof.** The action  $\theta$  is a morphism of mixed  $\mathbb{R}$ -Hodge structures. We understand them as bigraded objects equipped with the action of the Hodge Galois Lie algebra. In particular  $\theta$  preserves the bigrading and hence the weight.

The action  $\theta$  is determined by the images of the generators. The latter are operators of weight  $-1$  or  $-2$ . The weight  $-1$  generators are determined by the weight  $-1$  component of the connection  $\nabla_{\mathcal{G}}^0$  on  $\mathcal{L}_{\infty}$ . Indeed, the weight  $-1$  component of the parallel transform is determined by the weight  $-1$  component of the connection, and the same is true for the weight  $-1$  component of the exponential of the action of the generators. The weight  $-1$  component of our connection is given by the canonical 1-form  $\nu$  (Section 2.2), and the same is true for the standard one. Thus the weight  $-1$  generators act in the standard way.

Corollary 8.4 implies the claim for the weight  $-2$  generators  $X_s$ . This lemma is proved.

The Hodge Galois group acts by automorphisms, and hence acts trivially on the unit  $e$ . So there is an isomorphism of mixed  $\mathbb{R}$ -Hodge structures  $A^{\mathcal{H}}(X - S, a) \otimes e \xrightarrow{\sim} A(X - S, a)$ . Theorem 1.9 is proved.

The proof of Proposition 8.3 is a bit technical. Here is a crucial step.

**Dependence of  $\mathbf{G}_{v_a}$  on  $a$ .** Let us calculate the part of  $\mathbf{G}_{v_a}$  depending on the point  $a$ , and investigate what happens when  $a$  approaches to a puncture  $s$ . Recall (58) that

$$G_a(x_1, x_2) = G_{\mathrm{Ar}}(x_1, x_2) - G_{\mathrm{Ar}}(a, x_2) - G_{\mathrm{Ar}}(x_1, a).$$

**Lemma 8.6** *Assume that none of the vertices of an edge  $E$  of a decorated tree  $T$  is as on Fig 20, i.e. either  $S$ -decorated or decorated by a pair of 1-forms. Then replacing the Green function  $G_{v_a}$  assigned to  $E$  by  $G_{\text{Ar}}(x_1, x_2)$  we do not change the Hodge correlator integral assigned to  $T$ .*

**Proof.** Let  $v_1, v_2$  be the vertices of  $E$ . Each of them is a 3-valent vertex. Denote by  $x_1, x_2$  the points on the curves assigned to them. Let us assign the function  $G_{\text{Ar}}(x_1, a)$  to the edge  $E$ . Then the integral over  $x_2$  is proportional to one of the following:

$$\int_X d^{\mathbb{C}} G_a(y, x_2) \wedge d^{\mathbb{C}} G_a(z, x_2), \quad \text{or} \quad \int_X d^{\mathbb{C}} G_a(y, x_2) \wedge \alpha.$$

Each of them is zero: we integrate  $d^{\mathbb{C}}$ -exact forms. Similarly for  $G_{\text{Ar}}(a, x_2)$ . The lemma is proved.



Figure 20: Changing the Green function assigned to the edge  $E$  may alter the Hodge correlator.

Let  $\{\alpha_i, \alpha_j^{\vee}\}$  be a Darboux basis in the symplectic space  $\Omega_X^1 \oplus \overline{\Omega}_X^1$ , so that  $\int_X \alpha_i \wedge \alpha_j^{\vee} = \delta_{ij}$ . Let  $\{h_i, h_j^{\vee}\}$  be the dual basis in  $H_1(X, \mathbb{C})$ . Then the Hodge correlator is given by

$$\mathbf{G}_a = \sum_W \frac{1}{|\text{Aut}(W)|} \text{Cor}_{\mathcal{H}, a}(\beta_1 \otimes \dots \otimes \beta_m) \otimes \mathcal{C}(h_{\beta_1} \otimes \dots \otimes h_{\beta_m}), \quad \beta_i \in \{\alpha_i, \alpha_i^{\vee}\} \quad (159)$$

The sum is over a basis  $W = \mathcal{C}(\beta_1 \otimes \dots \otimes \beta_m)$  in the cyclic envelope of the tensor algebra of  $H^1(X, \mathbb{C})$ , and  $\{h_{\beta}\}$  is the dual basis to the basis  $\{\beta\}$ .

**Contribution of an edge  $E$  as on Fig 20.** Let us calculate the depending on the point  $a$  contribution to the integral of an edge  $E$  as on Fig 20. Take a tree  $T$  with such an edge  $E$ . Cut out the “small” tree growing from  $E$ , getting a new tree. Let  $W$  be its decoration inherited from  $T$ . For example, the decoration of the left tree on Fig 20 is  $\mathcal{C}(s \otimes W)$ . We assert that the depending on  $a$  part of the contribution of the edge  $E$  to the sum (159) is

$$\text{Cor}_{\mathcal{H}, \text{Ar}}(\{a\} \otimes W) \cdot \mathcal{C}(X_a \otimes W^{\vee}),$$

where  $W^{\vee}$  is the cyclic word in  $H_1$  dual to the cyclic words  $W$  in  $H^1$ . Indeed, take first the tree on the right of Fig 20, and assign to the edge  $E$  the Green function  $G_{\text{Ar}}(a, y)$ . Then its contribution to the integral is

$$\text{Cor}_{\mathcal{H}, \text{Ar}}(\{a\} \otimes W) \int_X \alpha \otimes \beta.$$

Substituting basis elements  $\alpha_i, \alpha_j^{\vee}$  for  $\alpha$  as well as for  $\beta$ , and taking the sums over  $i, j$ , we get

$$\text{Cor}_{\mathcal{H}, \text{Ar}}(\{a\} \otimes W) \cdot \mathcal{C}\left(\sum_i [h_i, h_i^{\vee}] \otimes W^{\vee}\right).$$

Similarly, the tree on the left on Fig 20 contributes

$$\text{Cor}_{\mathcal{H}, \text{Ar}}(\{a\} \otimes W) \cdot \mathcal{C}(\sum_{s \in S} X_s \otimes W^\vee).$$

Since  $X_a = \sum_{s \in S} X_s + \sum_i [h_i, h_i^\vee]$ , the total contribution is  $\text{Cor}_{\mathcal{H}, \text{Ar}}(\{a\} \otimes W) \cdot \mathcal{C}(X_a \otimes W^\vee)$ .

**Remark.** A similar argument shows that the Green operator induced on  $L(X - \{a\}, v_a)/(v_a)$  does not depend on the choice of a constant in the Green function  $G_a(x, y)$ . Indeed, we may assume that an edge  $E$  as on Fig 20 contributes to the integral the Green function but not its derivative. Then adding a constant to  $G_a(x, y)$  amounts to changing a factor in (159) by

$$\sum_{s \in S} X_s + \sum_{i=1}^g [h_i, h_i^\vee] = 0.$$

*Conclusion.* Since we may have several edges like on Fig 20, the depending on  $a$  part of  $\mathbf{G}_{v_a}$  is a sum of expressions

$$\text{Cor}_{\mathcal{H}, \text{Ar}}(\{a\} \otimes W_1 \otimes \dots \otimes \{a\} \otimes W_m) \cdot \mathcal{C}(X_a \otimes W_1^\vee \otimes \dots \otimes X_a \otimes W_m^\vee).$$

**Proof of Proposition 8.3.** Denote by  $\mathbf{G}_{p,q}$  the  $(p, q)$ -component of  $\mathbf{G}_{v_a}$ .

Let us investigate  $\mathbf{G}_{1,1}$ . There are only two cyclic words contributing to the depending on  $a$  part of  $\mathbf{G}_{1,1}$ :  $W' = \mathcal{C}(\{a\} \otimes \{s\})$  and  $W'' = \mathcal{C}(\{a\} \otimes \alpha \otimes \bar{\beta})$ . The first appears with the coefficient given by the Green function  $G_{\text{Ar}, v_a}(a, s)$ , and has the singularity  $\log t$ . It delivers the singular term  $\log t \cdot \text{ad}(X_s)$ . The second does not depend on  $s$ . So the claim about  $\mathbf{G}_{1,1}$  follows.

Let us show that  $\mathbf{G}_{p,q} \in \mathcal{A}_{\log}(X)$  for  $(p, q) = (1, q), (p, 1), (2, 2)$  provided that  $(p, q) \neq (1, 1)$ . After this the Lemma follows immediately by the induction using the differential equations on  $\mathbf{G}_{p,q}$ . Notice that since  $\bar{\mathbf{G}}_{1,q} = -\mathbf{G}_{q,1}$ , the claim about  $\mathbf{G}_{1,q}$  implies the one about  $\mathbf{G}_{q,1}$ .

Let us prove that  $\mathbf{G}_{1,q} \in \mathcal{A}_{\log}(X)$  for  $q > 1$ . The only cyclic words contributing to the depending on  $a$  part of  $\mathbf{G}_{1,q}$  are  $\mathcal{C}(\{a\} \otimes \{s\} \otimes \bar{\alpha}_1 \otimes \dots \otimes \bar{\alpha}_{q-1})$ , where  $\bar{\alpha}_i$  are antiholomorphic 1-forms on  $\bar{X}$ . The crucial case is  $q = 2$ . Let us show that  $\text{Cor}_{\mathcal{H}}(\{a\} \otimes \{s\} \otimes \bar{\alpha}) \in \mathcal{A}_{\log}(X)$ . We have to investigate the behaviour of the following integral at  $a \rightarrow s$ :

$$\int_{\bar{X}} G(a, x) \partial G(s, x) \wedge \bar{\alpha}.$$

Choose a small neighborhood  $U_s$  of  $s$ . We may assume that it contains  $a$ . Then the integral over its complement is clearly a smooth function in  $a$ . So we have to show that the integral over  $U_s$ , as a function of  $a \in U_s$ , has a tame logarithmic singularity at  $a = s$ . The problem is local. So it is sufficient to prove it for a single differential form  $\bar{\alpha}$  in  $U_s$  which does not vanish at  $s$ . So we may assume  $U_s$  is a small neighborhood of 0 in  $\mathbb{C}$ , and consider an integral

$$\int_{\mathbb{CP}^1} \log |x| \partial \log |x - a| \wedge \bar{\partial} \log |x - 1|.$$

This integral equals to a non-zero multiple of the Bloch-Wigner dilogarithm function  $\mathcal{L}_2(a)$ , see Section 10.1.2, which is independent of this Section. Since

$$d\mathcal{L}_2(a) = \log |1 - a| d \arg a - \log |a| d \arg(1 - a),$$

it has a tame logarithmic singularity at  $a = 0$ . This implies the claim. The claim for  $q > 2$  follows easily from the  $q = 2$  case.

Finally,  $\mathbf{G}_{2,2}$  is given by either  $\text{Cor}_{\mathcal{H}}(\{a\} \otimes \{s\} \otimes \bar{\alpha} \otimes \beta)$ , which reduces to the previous case, or by  $\text{Cor}_{\mathcal{H}}(\{a\} \otimes \{s\} \otimes \otimes \{x\})$ , which is completely similar to the  $\mathbf{G}_{1,2}$  case, and done by the reduction to the dilogarithm. Proposition 8.3 is proved.

## 9 Motivic correlators on curves

In Section 7 we described the Lie algebra of special derivations in the Betti realization. In Section 9 we present it in the motivic set-up. Using this we define certain elements of the motivic Lie coalgebra, *motivic correlators on curves*. Section 8 implies that the Hodge correlators on curves are their periods.

Let  $X$  be an irreducible regular projective curve over a field  $F$ , and  $S$  a finite subset of  $X(F)$ . We explore the motivic fundamental Lie algebra  $L(X - S, v_0)$  of the curve  $X - S$  with a tangential base point  $v_0$  over  $F$ .

We start by describing several different frameworks: Hodge,  $l$ -adic, motivic, etc. In each of them we supposed to have a Lie algebra in the corresponding category. It is equipped with the weight filtration  $W_{\bullet}$ , and  $\text{gr}^W L(X - S, v_0)$  is expressed via  $\text{gr}^W H^1(X - S)$ .

The  $l$ -adic realization  $L(X - S, v_0)$  is an  $l$ -adic pro-Lie algebra  $L^{(l)}(X - S, v_0)$  – the Lie algebra of the pro- $l$  completion  $\pi_1^{(l)}(X - S, v_0)$  of the fundamental group. The Galois group of  $F$  acts by automorphisms of  $\pi_1^{(l)}(X - S, v_0)$ , and hence by derivations of the Lie algebra  $L^{(l)}(X - S, v_0)$ , preserving some data, mostly related to the set  $S$ . We call such derivations *special*.

### 9.1 The motivic framework

Let  $F$  be a field. Below we work in one of the following five categories  $\mathcal{C}$ :

- i) *Motivic*. The hypothetical abelian category of mixed motives over a field  $F$ .
- ii) *Hodge*.  $F = \mathbb{C}$ , and  $\mathcal{C}$  is the category of mixed  $\mathbb{Q}$ - or  $\mathbb{R}$ -Hodge structures.
- iii) *Mixed  $l$ -adic*.  $F$  is an arbitrary field such that  $\mu_{l^\infty} \not\subset F$ , and  $\mathcal{C}$  is the mixed category of  $l$ -adic  $\text{Gal}(\bar{F}/F)$ -modules with a filtration  $W_{\bullet}$  indexed by integers, such that  $\text{gr}_n^W$  is a pure of weight  $n$ .
- iv) *Motivic Tate*.  $F$  is a number field,  $\mathcal{C}$  is the abelian category of mixed Tate motives over  $F$ , equipped with the Hodge and  $l$ -adic realization functors, c.f. [DG].
- v) *Variations of mixed Hodge structures*.  $F = \mathbb{C}$ , and  $\mathcal{C}$  is the category of mixed  $\mathbb{Q}$ - or  $\mathbb{R}$ -Hodge structures over a base  $B$ .

The setup i) is conjectural. The other four are well defined. A category  $\mathcal{C}$  from the list is an  $L$ -category, where  $L = \mathbb{Q}$  in i), iv), v);  $L = \mathbb{Q}$  or  $\mathbb{R}$  in ii); and  $L = \mathbb{Q}_l$  in iii).

Each of the categories has an invertible object, the *Tate object*, which is denoted, abusing notation,  $\mathbb{Q}(1)$  in all cases, although  $\text{Hom}_{\mathcal{C}}(\mathbb{Q}(1), \mathbb{Q}(1)) = L$ . So, for instance, in the setup iii)  $\mathbb{Q}(1)$  denotes the Tate module  $\mathbb{Q}_l(1)$ . We set  $\mathbb{Q}(n) := \mathbb{Q}(1)^{\otimes n}$ .

Each object in  $\mathcal{C}$  carries a canonical weight filtration  $W_{\bullet}$ , morphisms in  $\mathcal{C}$  are strictly compatible with this filtration. The weight of  $\mathbb{Q}(1)$  is  $-2$ .

An object  $M$  of  $\mathcal{C}$  is *pure* if  $\text{gr}_n^W M$  is zero for all  $n$  but possibly one. Tensor powers and direct summands of pure objects are pure. Let  $M$  be a pure object in  $\mathcal{C}$ . The pure subcategory



of  $\mathcal{C}$  generated by  $M$  is the smallest subcategory containing  $M$ , and closed under operations of taking tensor products, duals and direct summands. The mixed subcategory of  $\mathcal{C}$  generated by  $M$  is the smallest subcategory of  $\mathcal{C}$  closed under extensions and containing the pure subcategory generated by  $M$ .

Let us briefly recall some features of the Tannakian formalism in the setup of mixed categories which is used below, see [G3] for the proofs. See also [G7] for the mixed Tate case.

There is a canonical fiber functor to the pure tensor category  $\mathcal{P}^{\mathcal{C}}$  of pure objects in  $\mathcal{C}$ :

$$\omega : \mathcal{C} \longrightarrow \mathcal{P}^{\mathcal{C}}, \quad X \longmapsto \oplus_n \mathrm{gr}_n^W X.$$

The ind-object  $\mathrm{End}(\omega)$  of endomorphisms of this functor has a natural structure of a Hopf algebra in the tensor category  $\mathcal{P}^{\mathcal{C}}$ . Let  $\mathcal{A}_{\bullet}(\mathcal{C})$  be its graded dual. It is a commutative graded Hopf algebra. The functor  $\omega$  provides a canonical equivalence between the category  $\mathcal{C}$  and the category of graded  $\mathcal{A}_{\bullet}(\mathcal{C})$ -comodules in the category  $\mathcal{P}^{\mathcal{C}}$ . There is the a Lie coalgebra

$$\mathcal{L}_{\bullet}(\mathcal{C}) = \frac{\mathcal{A}_{\bullet}(\mathcal{C})}{\mathcal{A}_{>0}(\mathcal{C})^2}$$

with the cobracket induced by the coproduct in  $\mathcal{A}_{\bullet}(\mathcal{C})$ . Let  $\mathrm{L}_{\bullet}(\mathcal{C})$  be the dual Lie algebra.

**Definition 9.1** *Let  $X$  be a regular projective curve over  $F$ . Then  $\mathcal{P}_X$  (respectively  $\mathcal{C}_X$ ) is the subcategory of  $\mathcal{P}^{\mathcal{C}}$  (respectively  $\mathcal{C}$ ) generated by  $H^1(X)$ .*

Restricting the fiber functor  $\omega$  to  $\mathcal{C}_X$  we get an equivalence between the category  $\mathcal{C}_X$  and the category of graded  $\mathcal{A}_{\bullet}(\mathcal{C}_X)$ -comodules in  $\mathcal{P}_X$ . There is the corresponding Lie coalgebra

$$\mathcal{L}_{\bullet}(\mathcal{C}_X) = \frac{\mathcal{A}_{\bullet}(\mathcal{C}_X)}{\mathcal{A}_{>0}(\mathcal{C}_X)^2} \subset \mathcal{L}_{\bullet}(\mathcal{C})$$

and the dual Lie algebra  $\mathrm{L}_{\bullet}(\mathcal{C}_X)$ , a quotient of  $\mathrm{L}_{\bullet}(\mathcal{C})$ .

The Hopf algebra  $\mathcal{A}_{\bullet}(\mathcal{C})$  can be defined using the framed objects in  $\mathcal{C}$ , see [G3].

The fundamental Lie coalgebra  $\mathcal{L}_{\bullet}(\mathcal{C})$  can be written as a direct sum

$$\mathcal{L}_{\bullet}(\mathcal{C}) = \oplus_M \mathcal{L}_M(\mathcal{C}) \ominus M^{\vee} \tag{160}$$

over the set of all isomorphism classes of simple objects in  $\mathcal{C}$ .

## 9.2 $\mathcal{C}$ -motivic fundamental algebras of curves

Below we recall main features of the fundamental Lie/Hopf algebras of an irreducible smooth open curve in the motivic framework and in the realizations. The classical point of view is presented in Section 1.2 and Section 7. The  $l$ -adic one is recalled in the end.

Let  $S = \{s_0, \dots, s_n\} \subset X(F)$ , and  $v_0$  is a tangent vector at  $s_0 \in S$  defined over  $F$ . One should have a Hopf algebra  $A^{\mathcal{C}}(X - S, v_0)$  in the category  $\mathcal{C}$ , the  *$\mathcal{C}$ -motivic fundamental Hopf algebra of  $X - S$  with the tangential base point  $v_0$* . It was defined in the setups ii) and iii) in [D1], and in the setup iv) in [DG]. For the setup v) see [HZ]. Set

$$A_{X,S}^{\mathcal{C}} := \mathrm{gr}_{\bullet}^W A^{\mathcal{C}}(X - S, v_0).$$

We use notation  $A$  and  $\mathbb{A}$  for  $A^C(X - S, v_0)$  and  $\mathbb{A}_{X,S}^C$ . The tangential base point  $v_0$  gives rise to a canonical morphism, the “canonical loop around  $s_0$ ”:

$$X_{s_0} : \mathbb{Q}(1) \longrightarrow A.$$

The punctures  $s \in S^*$  provide conjugacy classes in  $A$ , corresponding to “loops around  $s$ ”. By passing to the associated graded we arrive at canonical morphisms

$$X_s : \mathbb{Q}(1) \longrightarrow \mathbb{A}, \quad s \in S^*.$$

A derivation  $\mathcal{D}$  of the algebra  $\mathbb{A}$  is called *special* if there exist objects  $B_s \subset \mathbb{A}$  such that

$$\mathcal{D}(X_s) = [B_s, X_s], \quad s \in S^*; \quad \text{and} \quad \mathcal{D}(X_{s_0}) = 0.$$

The formula on the left means that  $\mathcal{D}(X_s(\mathbb{Q}(1)))$  is isomorphic to the image of the commutator map applied to the object  $B_s \otimes X_s(\mathbb{Q}(1))$ . Special derivations form a Lie algebra, denoted by  $\text{Der}^S \mathbb{A}$ .

Set

$$\mathbb{S}^* = \mathbb{Q}[S^*] \otimes H_2(X), \quad \mathbb{H} = H_1(X); \quad \mathbb{V} := \mathbb{S}^* \oplus \mathbb{H} \cong \text{gr}^W H_1(X - S).$$

Let  $T_{\mathbb{V}}$  be the tensor algebra of  $\mathbb{V}$ .

**Lemma 9.2** *The Hopf algebra  $\mathbb{A}$  is isomorphic to the tensor algebra  $T_{\mathbb{V}}$ .*

**Proof.** The Hopf algebra (13) is identified with the Betti realization of the motivic fundamental Hopf algebra of  $X - S$ . This implies Lemma 9.2.

**An example:  $l$ -adic fundamental algebras.** Let  $\pi_1^{(l)} := \pi_1^{(l)}(X - S, v_0)$  be the pro- $l$  completion of the fundamental group of  $X - S$  with the tangential base point at  $v_0$ . It gives rise to a pronilpotent Lie algebra  $L^{(l)}(X - S, v_0)$  over  $\mathbb{Q}_l$  as follows. Let  $\pi_1^{(l)}(k)$  be the lower central series of  $\pi_1^{(l)}$ . Then  $\pi_1^{(l)}/\pi_1^{(l)}(k)$  is an  $l$ -adic Lie group, and

$$L^{(l)}(X - S, v_0) := \varprojlim \text{Lie}(\pi_1^{(l)}/\pi_1^{(l)}(k)).$$

It is the  $l$ -adic realization of the fundamental Lie algebra  $L(X - S, v_0)$ .

If  $X$  is defined over  $\mathbb{C}$ , the comparison theorem between the Betti and  $l$ -adic realizations gives rise to a Lie algebra isomorphism  $L^{(l)}(X - S, v_0) = \pi_1^{\text{nil}}(X(\mathbb{C}) - S, v_0) \otimes \mathbb{Q}_l$ .

If  $X$  is defined over a field  $F$ , the Galois group  $\text{Gal}(\overline{F}/F)$  acts by automorphisms of  $L^{(l)}(X - S, v_0)$ , giving rise to a canonical homomorphism

$$\text{Gal}(\overline{F}/F) \longrightarrow \text{Aut} L^{(l)}(X - S, v_0).$$

The tangent vector  $v_0$  provides a morphism of Galois modules, the “canonical loop around  $s_0$ ”:

$$X_{s_0} : \mathbb{Q}_l(1) \longrightarrow L^{(l)}(X - S, v_0).$$

Thus every puncture  $s \in S^*$  gives rise to a conjugacy class in  $L^{(l)}(X - S, v_0)$  preserved by the Galois group. An automorphism of  $L^{(l)}(X - S, v_0)$  is *special* if it preserves  $X_{s_0}(\mathbb{Q}_l(1))$  and the conjugacy classes around all other punctures. Denote by  $\text{Aut}^S L^{(l)}(X - S, v_0)$  the group of all special automorphisms of  $L^{(l)}(X - S, v_0)$ . We get a map

$$\text{Gal}(\overline{F}/F(\zeta_{l^\infty})) \longrightarrow \text{Aut}^S L^{(l)}(X - S, v_0).$$

### 9.3 The Lie algebra of special derivations in the motivic set-up

We use a shorthand

$$\mathcal{C}_{\mathbb{V}} := \frac{T_{\mathbb{V}}}{[T_{\mathbb{V}}, T_{\mathbb{V}}]}, \quad \mathcal{H}_X = \mathcal{H} := H^2(X).$$

We are going to define an action of  $\mathcal{C}_{\mathbb{V}} \otimes \mathcal{H}$  by special derivations of the algebra  $T_{\mathbb{V}}$ , and introduce a Lie algebra structure on  $\mathcal{C}_{\mathbb{V}} \otimes \mathcal{H}$ , making this action into an action of a Lie algebra.

Since  $T_{\mathbb{V}}$  is the free associative algebra in a semi-simple category generated by  $\mathbb{V}$ , there is a non-commutative differential map

$$\mathbb{D} : \mathcal{C}_{\mathbb{V}} \longrightarrow T_{\mathbb{V}} \otimes \mathbb{V}, \quad \mathbb{D}\mathcal{C}(a_0 \otimes a_1 \otimes \dots \otimes a_m) := \text{Cyc}_{m+1}(a_0 \otimes \dots \otimes a_{m-1}) \otimes a_m,$$

where  $\text{Cyc}_{m+1}$  means the cyclic sum.

Let us define a canonical map

$$\langle * \cap \mathcal{H} \cap * \rangle : \mathbb{V} \otimes \mathcal{H} \otimes \mathbb{V} \longrightarrow \mathbb{Q} \oplus \mathbb{S}^*. \quad (161)$$

Its only non-zero components are the following:

$$\langle * \cap \mathcal{H} \cap * \rangle_{\mathbb{H}} : \mathbb{H} \otimes \mathcal{H} \otimes \mathbb{H} \longrightarrow \mathbb{Q}, \quad \langle * \cap \mathcal{H} \cap * \rangle_{\mathbb{S}^*} : \mathbb{S}^* \otimes \mathcal{H} \otimes \mathbb{S}^* \longrightarrow \mathbb{S}^*. \quad (162)$$

The first one is the intersection pairing on  $H_1(X)$ . The second is given by  $\langle X_s \cap \mathcal{H} \cap X_t \rangle_{\mathbb{S}^*} = \delta_{st} X_s$ .

**An action of  $\mathcal{C}_{\mathbb{V}} \otimes \mathcal{H}$  by special derivations of  $T_{\mathbb{V}}$ .** We are going to define a map

$$\kappa : \mathcal{C}_{\mathbb{V}} \otimes \mathcal{H} \longrightarrow \text{Der}^S(T_{\mathbb{V}}). \quad (163)$$

Since the algebra  $T_{\mathbb{V}}$  is free, to define its derivation we just define it on the generators, i.e. produce an element in  $T_{\mathbb{V}} \otimes \mathbb{V}^{\vee}$ . We get it as a composition

$$\mathcal{C}_{\mathbb{V}} \otimes \mathcal{H} \xrightarrow{\mathbb{D} \otimes \text{Id}} T_{\mathbb{V}} \otimes (\mathbb{H} \oplus \mathbb{S}^*) \otimes \mathcal{H} \xrightarrow{(162)^*} T_{\mathbb{V}} \otimes (\mathbb{H}^{\vee} \oplus \mathbb{S}^* \otimes \mathbb{S}^{*\vee}) = T_{\mathbb{V}} \otimes \mathbb{H}^{\vee} \bigoplus \mathbb{S}^* \otimes T_{\mathbb{V}} \otimes \mathbb{S}^{*\vee}$$

followed by the commutator map  $\mathbb{S}^* \otimes T_{\mathbb{V}} \longrightarrow T_{\mathbb{V}}$ . The second arrow is the dual to the map (162).

**A Lie bracket on  $\mathcal{C}_{\mathbb{V}} \otimes \mathcal{H}$ .** There is a canonical map  $\theta : \mathcal{C}_{\mathbb{V}} \otimes \mathcal{H} \otimes \mathcal{C}_{\mathbb{V}} \longrightarrow T_{\mathbb{V}}$  given as

$$\mathcal{C}_{\mathbb{V}} \otimes \mathcal{H} \otimes \mathcal{C}_{\mathbb{V}} \xrightarrow{\mathbb{D} \otimes \text{Id} \otimes \mathbb{D}} T_{\mathbb{V}} \otimes \mathbb{V} \otimes \mathcal{H} \otimes \mathbb{V} \otimes T_{\mathbb{V}} \longrightarrow T_{\mathbb{V}} \otimes T_{\mathbb{V}} \otimes \mathbb{V} \otimes \mathcal{H} \otimes \mathbb{V} \xrightarrow{(161)} T_{\mathbb{V}} \otimes T_{\mathbb{V}} \otimes (\mathbb{Q} \oplus \mathbb{S}^*) \longrightarrow T_{\mathbb{V}}.$$

The last map is the product for the  $\mathbb{Q}$ -component, and the commutator map  $[\cdot, \cdot] : T_{\mathbb{V}} \otimes T_{\mathbb{V}} \rightarrow T_{\mathbb{V}}$  followed by the product with  $\mathbb{S}^*$  for the second component. We define a Lie bracket as follows:

$$\{F \otimes \mathcal{H}, G \otimes \mathcal{H}\} := \mathcal{C} \circ \theta \left( F \otimes \mathcal{H} \otimes G - G \otimes \mathcal{H} \otimes F \right) \otimes \mathcal{H} \quad (164)$$

**Explicit formulas.** Let  $\mathbb{H} = \oplus_i p_i$  be a decomposition into simple objects. Then

$$\begin{aligned} \mathbb{D}F &= \sum_i \frac{\partial F}{\partial p_i} \otimes p_i + \sum_{s \in S^*} \frac{\partial F}{\partial X_s} \otimes X_s. \\ \kappa : p &\mapsto \frac{\partial F}{\partial q} \langle q \cap \mathcal{H} \cap p \rangle, \quad X_{s_0} \mapsto 0, \quad X_s \mapsto \left[ X_s, \frac{\partial F}{\partial X_s} \right], \quad s \in S^*. \\ \theta : F \otimes \mathcal{H} \otimes G &\mapsto \sum_{i,j} \frac{\partial F}{\partial p_i} \otimes \frac{\partial G}{\partial p_j} \otimes \langle p_i \cap \mathcal{H} \cap p_j \rangle + \sum_{s \in S^*} \left[ \frac{\partial F}{\partial X_s}, \frac{\partial G}{\partial X_s} \right] \otimes \langle X_s \cap \mathcal{H} \cap X_s \rangle. \\ \{F \otimes \mathcal{H}, G \otimes \mathcal{H}\} &= 2 \sum_{s \in S^*} \mathcal{C} \left( \left[ \frac{\partial F}{\partial X_s}, \frac{\partial G}{\partial X_s} \right] \otimes X_s \right) \otimes \mathcal{H} + \\ &\quad \sum_{p,q} \langle p \cap \mathcal{H} \cap q \rangle \otimes \mathcal{C} \left( \frac{\partial F}{\partial p} \otimes \frac{\partial G}{\partial q} - \frac{\partial G}{\partial p} \otimes \frac{\partial F}{\partial q} \right) \otimes \mathcal{H}. \end{aligned} \quad (165)$$

Just like in Section 7, we define  $\mathcal{C}'_{\mathbb{V}}$  as the quotient of  $\mathcal{C}_{\mathbb{V}}$  by  $\oplus_{s \in S^*} \mathbb{Q}[X_s]$ .

**Theorem 9.3** *The map  $\kappa$  is a morphism of Lie algebras. It induces an isomorphism*

$$\kappa' : \mathcal{C}'_{\mathbb{V}} \otimes \mathcal{H} \xrightarrow{\sim} \text{Der}^S(\mathbb{T}_{\mathbb{V}}). \quad (166)$$

*The map  $\kappa$  provides an isomorphism of Lie algebras*

$$\kappa : \mathcal{CLie}_{\mathbb{V}} \otimes \mathcal{H} \xrightarrow{\sim} \text{Der}^S(\mathbb{L}_{\mathbb{V}}). \quad (167)$$

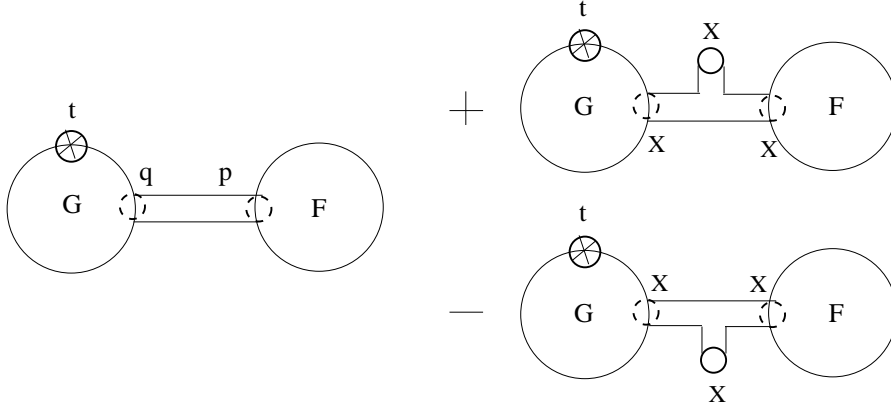


Figure 21: Calculation of  $\kappa_F \circ \kappa_G(h)$ .

**Proof.** Let us show that the map (166) respects the brackets. Let us calculate how the commutator  $[\kappa_{F \otimes \mathcal{H}}, \kappa_{G \otimes \mathcal{H}}]$  acts on  $h \in \mathbb{H}$ . To calculate the  $\mathbb{H}$ -part of the action, we have to insert  $\partial F / \partial p$  instead of each factor  $q$  in  $\partial G / \partial t$  – this operation is denoted by  $\frac{q}{\partial}$ . We get:

$$\kappa_{F \otimes \mathcal{H}} \circ \kappa_{G \otimes \mathcal{H}} : h \mapsto \sum_t \kappa_{F \otimes \mathcal{H}} \left( \frac{\partial G}{\partial t} \right) \otimes \langle t \cap \mathcal{H} \cap h \rangle =$$

$$\sum_t \left( \sum_{p,q} \frac{\partial F}{\partial p} \xrightarrow{q} \frac{\partial G}{\partial t} \otimes \langle p \cap \mathcal{H} \cap q \rangle + \sum_{s \in S^*} \left[ \frac{\partial F}{\partial X_s}, \frac{\partial G}{\partial X_s} \right] X_s \right) \otimes \langle t \cap \mathcal{H} \cap h \rangle.$$

The (left) sums are over bases  $\{t\}, \{p\}, \{q\}$  in  $\mathbb{H}$ . The result is visualized on Fig 21. Big circles show the cyclic words  $F$  and  $G$ ; little circles show the variables added, and the punctured little circles show the variables removed in the process of calculation. The added / removed factors shown nearby. The bridge between big circles shows how we make the product. The special role of  $t$  is emphasized by a cross. It is clear from this that  $[\kappa_{F \otimes \mathcal{H}}, \kappa_{G \otimes \mathcal{H}}]$  acts on  $h$  just like the commutator  $\kappa_{\{F \otimes \mathcal{H}, G \otimes \mathcal{H}\}}$ . The action on  $X_s$  is treated similarly – the only difference is that we have to replace the operation of removing of  $t$  by the commutator with  $X_s$ .

The rest follows from this and Proposition 7.4 by going to the Betti realization, or repeating the argument in the motivic set-up. The theorem is proved.

## 9.4 Motivic correlators on curves

We define three versions of the motivic correlators on curves, which differ in our treatment of the base points: base point motivic correlators, symmetric motivic correlators, and averaged base point motivic correlators.

**1. Base point motivic correlators on curves.** Since the fundamental Lie algebra  $L(X - S, v_0)$  is a pro-object in the category  $\mathcal{C}_X$ , it gives rise to a map

$$L_\bullet(\mathcal{C}_X) \longrightarrow \text{Der}^S \mathbb{L}_{X, S^*}. \quad (168)$$

Identifying Lie algebras  $\text{Der}^S \mathbb{L}_{X, S^*}$  and  $\mathcal{C}\mathcal{L}ie_{X, S^*} \otimes H_2(X)$  via isomorphism (167), and dualizing map (168), we arrive at a Lie coalgebra map

$$\Psi_{v_0} : \mathcal{C}\mathcal{L}ie_{X, S^*}^\vee \otimes H^2(X) \longrightarrow \mathcal{L}_\bullet(\mathcal{C}_X). \quad (169)$$

Abusing notation, write  $X_s^\vee$  for  $X_s(\mathbb{Q}(1))^\vee$ . There is an isomorphism

$$\Phi : \bigoplus_{m=1}^{\infty} \bigoplus_{a_i \in S^*} \mathbb{A}_{\mathbb{H}^\vee}^\vee \otimes X_{a_0}^\vee \otimes \dots \otimes X_{a_m}^\vee \otimes \mathbb{A}_{\mathbb{H}^\vee} \xrightarrow{\sim} \mathbb{A}_{\mathbb{V}^\vee}, \quad (170)$$

We denote by  $\Phi_{\text{cyc}}$  the map  $\Phi$  followed by the projection to the cyclic envelope of  $\mathbb{A}_{\mathbb{V}^\vee}$ . Here is our first definition, providing an element of the motivic Lie coalgebra:

**Definition 9.4** *Given points  $a_i \in S^*$  and simple summands  $M_0, \dots, M_m$  of  $\mathbb{A}_{\mathbb{H}^\vee}$ , the base point motivic correlator*

$$\text{Cor}_{X; v_0} \left( \{a_0\} \otimes M_0 \otimes \{a_1\} \otimes M_1 \otimes \dots \otimes \{a_m\} \otimes M_m \right) (1) \subset \mathcal{L}_w(\mathcal{C}_X) \quad (171)$$

*is the image of  $\mathcal{C} \left( X_{a_0}^\vee \otimes M_0 \otimes \dots \otimes X_{a_m}^\vee \otimes M_m \right) (1)$  under the map  $\Psi_{v_0} \circ \Phi_{\text{cyc}}$ . Here  $w = 2m + \sum \text{wt}(M_i)$ .*

The object (171) lies in the  $M_1^\vee \otimes \dots \otimes M_m^\vee(m)$ -isotypical component of  $\mathcal{L}_\bullet(\mathcal{C}_X)$  (see (160)).

Using description of the Hopf algebra  $\mathcal{A}_\bullet(\mathcal{C}_X)$  via minimal framed objects in  $\mathcal{C}_X$  ([G3]), and a similar description of the Lie coalgebra  $\mathcal{L}_\bullet(\mathcal{C}_X)$ , we conclude that the object (171) provides a minimal framed object in  $\mathcal{C}_X$ , well-defined modulo products of non-trivial objects in  $\mathcal{C}_X$ .

**2. Symmetric motivic correlators on curves.** Let us employ the symmetric description of the Lie algebra of special derivations (Section 7.3) in the motivic set-up. There is a Lie algebra  $\overline{\mathcal{C}}_{X,S}$  and its Lie subalgebra  $\overline{\mathcal{CLie}}_{X,S}$  in the  $\mathcal{C}$ -motivic category, whose Betti realizations are the Lie algebras  $\mathcal{CLie}(\overline{A}_{X,S})$  and  $\mathcal{CLie}(\overline{L}_{X,S})$ . Denote by  $\overline{\mathcal{C}}_{X,S}^\vee$  and  $\overline{\mathcal{CLie}}_{X,S}^\vee$  the dual Lie coalgebras. There is a Lie coalgebra map

$$\Psi^{\text{sym}} : \overline{\mathcal{CLie}}_{X,S}^\vee \otimes H^2(X) \longrightarrow \mathcal{L}_\bullet(\mathcal{C}_X). \quad (172)$$

**Definition 9.5** A symmetric motivic correlator is the image of  $W \in \overline{\mathcal{CLie}}_{X,S}^\vee(1)$  under map (172):

$$\text{Cor}_X^{\text{sym}}(W) := \Psi^{\text{sym}}(W) \in \mathcal{L}(\mathcal{C}_X).$$

The Lie coalgebras in question are related as follows:

$$\overline{\mathcal{CLie}}_{X,S}^\vee \longleftarrow \overline{\mathcal{C}}_{X,S}^\vee \hookrightarrow \mathcal{C}_{X,S}^\vee.$$

The left map is an epimorphism. In general there seems to be no simple description of the subspace  $\overline{\mathcal{C}}_{X,S}^\vee$  in  $\mathcal{C}_{X,S}^\vee$ . However when  $X$  is either a rational or an elliptic curve, such a description exists.

**Lemma 9.6** Let  $X$  be a rational curve. Then an element

$$\sum_{a_i \in S} c_{a_0, \dots, a_m} \mathcal{C}(X_{a_0}^\vee \otimes \dots \otimes X_{a_m}^\vee) \in \mathcal{C}_{P^1, S}^\vee \quad (173)$$

lies in the subspace  $\overline{\mathcal{C}}_{P^1, S}^\vee$  if and only if  $\sum_{a_k \in S} c_{a_0, \dots, a_m} = 0$  for any  $k = 0, \dots, m$ .

**Proof.** Elements (173) for arbitrary coefficients  $c_{a_0, \dots, a_m} \in \mathbb{Q}$  give a basis in  $\mathcal{C}_{P^1, S}^\vee$ . The orthogonality to the two-sided ideal generated by  $\sum_{s \in S} X_s$  is precisely the condition on the coefficients.

**Lemma 9.7** Let  $E$  be an elliptic curve,  $\mathbb{H}^\vee = H^1(E)$ . Then the subspace  $\overline{\mathcal{C}}_{E, S}^\vee$  consists of elements

$$\sum_{a_i \in S} c_{a_0, n_0; \dots, a_m, n_m} \mathcal{C}(X_{a_0}^\vee \otimes S^{n_0} \mathbb{H}^\vee \otimes \dots \otimes X_{a_m}^\vee \otimes S^{n_m} \mathbb{H}^\vee) \in \mathcal{C}_{E, S}^\vee, \quad (174)$$

such that for any  $k = 0, \dots, m$  one has  $\sum_{a_k \in S} c_{a_0, n_0; \dots, a_m, n_m} = 0$ .

**Proof.** We spell the argument in the Betti realization. So  $\mathbb{H}^\vee$  has a symplectic basis  $p, q$ . A basis of  $\mathcal{C}_{E, S}^\vee$  is given by cyclic words  $\mathcal{C}(a_0 \otimes F_0(p, q) \otimes \dots \otimes a_m \otimes F_m(p, q))$ , where  $a_i \in S$  and  $F_i(p, q)$  are non-commutative polynomials in  $p, q$ . The defining ideal is generated by the element  $[p, q] + \sum_{a \in S} X_a$ . So expressing  $[p, q]$  as  $-\sum_{a \in S} X_a$  we can replace the polynomials  $F_i(p, q)$  by commutative polynomials  $G_i(p, q)$ . We may assume that  $G_i$  are homogeneous of degree  $n_i$ . Since  $G_i(p, q)$  are commutative polynomials, they are orthogonal to  $[p, q]$ . So the orthogonality to the element  $[p, q] + \sum_{a \in S} X_a$  boils down to the orthogonality to  $\sum_{a \in S} X_a$ . The lemma is proved.

**Remark.** The symmetric motivic correlator map does not depend on the choice of the base point. It has the same image in the motivic Lie coalgebra as the one defined by using a base point. This apparent “contradiction” is resolved as follows. A choice of the base point provides a way to choose elements in the image of the motivic correlator map, without changing the latter. Different base points provide different representations, built using different supplies of elements in the same space. For example, as we see in Section 9.5.1, the elements of the Jacobian  $J_X \otimes \mathbb{Q}$  of  $X$  provided by the symmetric motivic correlators are  $s_i - s_j$ . The ones provided by the motivic correlators corresponding to a tangential base point at  $s_0$  are  $s_i - s_0$ . These are two different collection of elements of  $J_X \otimes \mathbb{Q}$ ; however they span the same subspace.

Similarly the symmetric Hodge correlators are complex numbers defined by the curve  $X - S$ . The  $\mathbb{Q}$ -vector space spanned by them does not depend on the choice of the base point, or a Green function. However choosing different base points we get different real MHS’s – the latter depend not only on the  $\mathbb{Q}$ -vector subspace spanned by the periods, but also on the way we view its elements as the matrix coefficients.

**3. Averaged base point motivic correlators.** Let us choose for every point  $s \in S$  a tangent vector  $v_s$  at  $s$ . There are natural choices for the elliptic and modular curves.

Recall the canonical morphism of Lie coalgebras assigned to a tangential base point  $v_s$  at  $s$ :

$$\text{Cor}_{X, S-\{s\}; v_s} : \mathcal{CT}(\mathbb{H}_{X, S-\{s\}}^\vee) \otimes H_2(X) \longrightarrow \mathcal{L}_{\text{Mot}}. \quad (175)$$

The collection of maps (175), for different choices of  $s \in S$ , is organized into a Lie coalgebra map

$$\bigoplus_{s \in S} \mathcal{CT}(\mathbb{H}_{X, S-\{s\}}^\vee) \otimes H_2(X) \longrightarrow \mathcal{L}_{\text{Mot}}. \quad (176)$$

Here on the left stands a direct sum of the Lie coalgebras. It is handy extend  $S - \{s\}$  to  $S$ , and enlarge the Lie coalgebra on the left to a Lie coalgebra

$$\mathcal{CT}(\mathbb{H}_{X, S}^\vee) \otimes H_2(X) \otimes \mathbb{Q}[S], \quad (177)$$

where the added elements have the coproduct zero. So the Lie coalgebra on the left of (176) is the quotient of (177). The composition (177)  $\rightarrow$  (176) provides a Lie coalgebra map

$$\text{Cor}_{X-S}^{\text{av}} : \mathcal{CT}(\mathbb{H}_{X, S}^\vee) \otimes H_2(X) \otimes \mathbb{Q}[S] \longrightarrow \mathcal{L}_{\text{Mot}}, \quad (178)$$

so that map (175) is the component of map (178) assigned to the factor  $\{s\} \in \mathbb{Q}[S]$ .

The averaged base point motivic correlator map arises when we take  $\frac{1}{|S|} \sum_{s \in S} \{s\} \in \mathbb{Q}[S]$ . It allows to work in towers of curves, see Section 11.2

## 9.5 Examples of motivic correlators

**1. The Jacobian of  $X$ .** Let  $J_X(F)$  be the Jacobian of the curve  $X$ . It is well known that in the motivic set-up (i) one should have

$$\text{Ext}_{\text{Mot}}^1(\mathbb{Q}(0), \mathbb{H}) = \mathcal{L}_1(\mathcal{C}_X) = J_X(F) \otimes \mathbb{Q}.$$

The motivic correlator  $\text{Cor}_{X, s_0}(\{a\} \otimes \mathbb{H}^\vee)$ , see Fig. 22, is given by the point  $\{a\} - \{s_0\}$  in the Jacobian  $J_X(F)$ . Notice the importance of the base point  $s_0$ . In the symmetric description the correlators are

$$\text{Cor}_X^{\text{sym}} \mathcal{C}((\{a\} - \{b\}) \otimes \mathbb{H}^\vee) = a - b \in J_X \otimes \mathbb{Q}.$$

They are defined only on degree zero divisors. In any case we get the same supply of elements of the motivic Lie coalgebra. In the  $l$ -adic set-up (iii) the target group is  $J_X(F) \otimes \mathbb{Q}_l$ .



Figure 22: The motivic correlator for  $\mathcal{C}(\{s_0\} \otimes \mathbb{H}^\vee)$  is the point  $s - s_0$  of the Jacobian of  $X$ .

**2. The rational curve case.** Let  $X = P^1$ , the set  $S$  includes  $\{0\} \cup \{\infty\}$ , and  $v = \partial/\partial t$  is the canonical vector at 0. Denote by  $dt/t$  the cohomology class given by the form  $dt/t$  in the De Rham realization. The motivic correlators in this case are

$$\mathbb{L}_{n_0, \dots, n_m}^{\mathcal{M}}(a_0; \dots; a_m) := \text{Cor}_{P^1; v} \left( \{a_0\} \otimes \{0\}^{\otimes n_0} \otimes \dots \otimes \{a_m\} \otimes \{0\}^{\otimes n_m} \right) \subset \mathcal{L}_w(\mathcal{C}_T). \quad (179)$$

Here  $a_i \in S^*$ ,  $w = 2(n_0 + \dots + n_m + m)$  is the weight, and  $m$  is the depth (Section 9.5.9). We call them the *cyclic motivic multiple polylogarithms*. Their periods coincide with the corresponding multiple polylogarithms up to the lower depth terms. In Section 10.1.4 we derive an integral formula expressing their real periods as Hodge correlator type integrals of the classical polylogarithms.

**Remark.** In [G4], given an abelian group  $G$ , we defined dihedral Lie coalgebras, which are generated, as vector spaces, by symbols  $L_{n_0, \dots, n_m}(g_0; \dots; g_m)$ ,  $g_i \in G$ . When  $G = \mathbb{C}^*$ , we related them to multiple polylogarithms, although to achieve this we had to work modulo the depth filtration. It is now clear that those generators match the motivic elements (179), and so their periods are the corresponding Hodge correlators rather than the single valued versions of the multiple polylogarithms. See Section 10 for more information.

An especially interesting case is when  $S = \mu_N \cup \{0\} \cup \{\infty\}$ . In this case we call the elements (179) the depth  $m$  motivic multiple Dirichlet  $L$ -values. For instance when  $S = \{0\} \cup \{1\} \cup \{\infty\}$ , we get the depth  $m$  motivic multiple  $\zeta$ -values.

**3. The elliptic curves case.** Let  $X = E$  be an elliptic curve over  $F$ . If  $E$  is not CM, the objects  $S^k \mathbb{H}$  are simple. The *symmetric motivic multiple elliptic polylogarithms* are defined as

$$\text{Cor}_{E; n_0, \dots, n_m}^{\text{sym}}(D_0; \dots; D_m) := \text{Cor}_E \left( D_0 \otimes S^{n_0} \mathbb{H}^\vee \otimes \dots \otimes D_m \otimes S^{n_m} \mathbb{H}^\vee \right) (1) \subset \mathcal{L}_w(\mathcal{C}_E). \quad (180)$$

Here  $D_i$  are degree zero divisors supported on  $S$  and  $w = n_0 + \dots + n_m + 2m$ . When  $E$  is a complex elliptic curve, their periods are given by the generalized Eisenstein-Kronecker series (Section 10.2).

**4. Motivic correlators of torsion points on elliptic curves.** Let  $X = E$  be an elliptic curve, and  $S = E[N]$  is the set of its  $N$ -torsion points. There is an almost canonical choice  $v_\Delta$  of the tangent vector at 0: its dual is given by 12-th root of the section of  $\Omega_{\mathcal{E}/\mathcal{M}}^{12}$ , where  $\mathcal{E}$  is the universal elliptic curve over the modular curve  $\mathcal{M}$ , given by the modular form  $\Delta = q \prod (1 - q^n)^{24}$ . We employ the corresponding  $E$ -invariant vector field  $v_\Delta$  on  $E$ , and assign to every missing point on  $E - E[N]$  the tangent vector  $\frac{1}{N} v_\Delta$ . We arrive at the averaged base point motivic correlator map. It is obviously  $E[N]$ -invariant, and thus descends to a map of  $E[N]$ -coinvariants:

$$\text{Cor}_{E-E[N]}^{\text{av}} : \left( \mathcal{CT}(\mathbb{H}_{E, E[N]}^\vee) \otimes \text{Meas}(E[N])(1) \right)_{E[N]} \longrightarrow \mathcal{L}_{\text{Mot}}.$$



There is a unique  $E[N]$ -invariant volume 1 measure  $\mu_E^0$  on  $E[N]$ . So we get a canonical map

$$\mathrm{Cor}_{E-E[N]}^0 : \mathcal{CT}(\mathbb{H}_{E,E[N]}^\vee)(1)_{E[N]} \otimes \mu_E^0 \longrightarrow \mathcal{L}_{\mathrm{Mot}}.$$

These are the motivic correlators with the “averaged tangent vector” at the  $N$ -torsion points.

**5. Motivic double elliptic logarithms at torsion points.** They were defined and studied in [G10]. Here is an alternative definition, coming from the action of the Galois group on the motivic fundamental group of an elliptic curve minus the torsion points. We use a shorthand

$$\theta_E(a_0 : \dots : a_n) := \mathrm{Cor}_E^0(\{a_0\} \otimes \dots \otimes \{a_n\})(1), \quad a_i \in E_{\mathrm{tors}}.$$

It agrees with the notation in [G10]. It is translation invariant.

Here is an  $l$ -adic / Hodge version of formula (58) expressing the Green function  $G_a(x, y)$  via the Arakelov Green function. We spell it in the  $l$ -adic set-up.

**Lemma 9.8** *For any  $x \in E(F)$ , and any  $E$ -invariant vector field  $v$  on  $E$  one has*

$$\theta_x(a_0 : a_1) = \theta_E(a_0 : a_1) - \theta_E(x : a_1) - \theta_E(a_0 : x) + C, \quad C \in \mathrm{Ext}_C^1(\mathbb{Q}_l(0), \mathbb{Q}_l(2)).$$

**Proof.** Indeed,

$$\delta\theta_x(a_0 : a_1) = \mathrm{Cor}_{E,x}(\{a_0\} \otimes \mathbb{H}^\vee)(1) \wedge \mathrm{Cor}_{E,x}(\{a_1\} \otimes \mathbb{H}^\vee)(1) = (a_0 - x) \wedge (a_1 - x) \in \Lambda^2 E(F).$$

The coproduct of the right hand side modulo  $N$ -torsion is the same. Therefore the difference lies in  $\mathrm{Ext}_C^1(\mathbb{Q}_l(0), \mathbb{Q}_l(2))$ . The latter  $l$ -adic Ext-group is rigid. Thus, as a variation in  $x, a_0, a_1$ , it is constant variation. The lemma is proved.

The constant  $C$  for the canonical vector  $v_\Delta$  is zero. Indeed,  $v_\Delta$ , viewed as a section over the modular curve, is defined over  $\mathbb{Q}$ , and we can take  $E, x, a_i$ 's defined over  $\mathbb{Q}$ , and use the fact that  $C$  is of geometric origin, and  $K_3^{\mathrm{ind}}(\mathbb{Q}) = 0$ .

**Lemma 9.9** *Assume that  $a_i \in E[N]$ . Then, in the  $l$ -adic or Hodge set-ups, one has*

$$\delta\theta_E(a_0 : a_1 : a_2) = \mathrm{Cycle}_{\{0,1,2\}} \left( \theta_E(a_0 : a_1) \wedge \theta_E(a_1 : a_2) \right) \quad \text{modulo } N\text{-torsion.} \quad (181)$$

**Proof.** We have  $N\delta_{\mathrm{Cas}}\theta_E(a_0 : a_1 : a_2) = 0$ . Indeed,

$$N\delta_{\mathrm{Cas}}\theta_E(a_0 : a_1 : a_2) = N\mathrm{Cycle}_{\{0,1,2\}} \left( \mathrm{Cor}_E^0(\{a_0\} \otimes \mathbb{H}^\vee)(1) \wedge \mathrm{Cor}_E^0(\mathbb{H}^\vee \otimes \{a_1\} \otimes \{a_2\})(1) \right),$$

and  $N\mathrm{Cor}_E^0(\{a_0\} \otimes \mathbb{H}^\vee) = Na_0 = 0$ . Using Lemma 9.8,  $\delta_S\theta_E(a_0 : a_1 : a_2)$  equals

$$\mathrm{Cycle}_{\{0,1,2\}} \frac{1}{N^2} \sum_{x \in E[N]} \theta_x(a_0 : a_1) \wedge \theta_x(a_0 : a_2) = \mathrm{Cycle}_{\{0,1,2\}} \left( \theta_E(a_0 : a_1) \wedge \theta_E(a_0 : a_2) \right).$$

The lemma is proved.

The elements  $\theta_E(a_0 : a_1 : a_2)$  from [G10] have the same coproduct, and thus coincide with the ones above up to an element of  $\mathrm{Ext}_C^1(\mathbb{Q}_l(0), \mathbb{Q}_l(2))$ . Each of them is skew-symmetric in  $a_0, a_1, a_2$ . Since the difference between them is constant on the modular curve, this implies that it is zero.

**6. Motivic correlators of torsion points on a CM elliptic curve.** Suppose that  $E$  is a CM curve. Then it has a complex multiplication by an order in the ring of integers in an imaginary quadratic field  $K$ . Extending the scalars from  $L$  to  $L \otimes K$ , we have a decomposition  $\mathbb{H}^\vee = \psi \oplus \bar{\psi}$  into a sum of pure motives corresponding to the two Hecke Grössencharacters. Set  $\psi^n := \psi^{\otimes n}$  and  $\bar{\psi}^n := \bar{\psi}^{\otimes n}$ . Then  $S^n \mathbb{H}^\vee = \oplus_{n'+n''=n} \psi^{n'} \bar{\psi}^{n''}$ . So we decompose the motivic correlators accordingly.

Let  $a_i$  be torsion points on  $E$ , and  $(n'_i, n''_i)$  non-negative integers. The corresponding *motivic Hecke Grössencharacters multiple L-values*  $L_{E; n'_0, n''_0; \dots; n'_m, n''_m}(a_0 : \dots : a_m)$  are motivic correlators

$$\text{Cor}_E^0 \left( \{a_0\} \otimes \psi^{n'_0} \bar{\psi}^{n''_0} \otimes \{a_1\} \otimes \psi^{n'_1} \bar{\psi}^{n''_1} \otimes \dots \otimes \{a_m\} \otimes \psi^{n'_m} \bar{\psi}^{n''_m} \right) (1).$$

**7. The Fermat curves case.** Let  $\mathbb{F}_N$  be the projective Fermat curve  $x^N + y^N = z^N$ . Let  $S$  be the intersection of  $\mathbb{F}_N$  with the coordinate triangle. The group  $\mu_N^2 = \mu_N^3 / \mu_N$  acts on the curve  $\mathbb{F}_N$ . The motive  $H^1(\mathbb{F}_N) \otimes \mathbb{Z}[\zeta_N]$  is decomposed into a direct sum of rank one motives  $\psi_\chi$  parametrized by the characters  $\chi$  of  $\mu_N^2$ . These are Weil's Jacobi sums Grössencharacter motives.

We use the symmetric motivic correlators. Just like in the CM elliptic curve case, by using the motives  $\psi_\chi$  and their powers, we get symmetric motivic correlators parametrised by cyclic tensor products of  $\psi_\chi^n$  and the Tate motives assigned to the points of  $S$ .

**8. The modular curves case.** This is one of the most interesting cases, see Section 11.

**9. The depth filtration.** Suppose that  $s_0 \in S' \subset S \subset X$ . Then the inclusion  $X - S \subset X - S'$  provides a projection of algebras  $A(X - S, v_0) \longrightarrow A^C(X - S', v_0)$ . A *depth filtration* on the Hopf algebra  $A(X - S, v_0)$  is a filtration by powers of its kernel. One defines similarly a depth filtration on the Lie algebra  $L(X - S, v_0)$ . It is induced by the embedding  $L(X - S, v_0) \hookrightarrow A(X - S, v_0)$ . Here are two examples.

(i) Assume that  $X \neq \mathbb{P}^1$ . The embedding  $X - S \hookrightarrow X$  provides a canonical projection

$$A(X - S, v_0) \longrightarrow A(X, s_0). \quad (182)$$

Let  $I_{X,S}$  be its kernel. A depth filtration  $D$  on the Hopf algebra  $A(X - S, v_0)$  is the filtration  $A(X - S, v_0) \supset I_{X,S} \supset I_{X,S}^2 \supset \dots$  by powers of the ideal  $I_{X,S}$ , indexed by  $m = 0, 1, 2, \dots$ . One has

$$\text{Gr}_m^D A(X - S, v_0) = \oplus_{s_i \in S^*} A_X \otimes X_{s_0} \otimes A_X \otimes \dots \otimes X_{s_m} \otimes A_X.$$

(ii) The depth filtration for  $\mathbb{G}_m - S$  is defined using the embedding  $\mathbb{G}_m - S \hookrightarrow \mathbb{G}_m$ .

## 10 Examples of Hodge correlators

### 10.1 Hodge correlators on $\mathbb{CP}^1 - S$ and polylogarithms

**1. Multiple Green functions.** Let  $a_0, \dots, a_m$  be points of a complex curve  $X$ . Let  $\mu$  be a volume one measure on  $X$ . The Hodge correlator of a cyclic word  $\mathcal{C}(\{a_0\} \otimes \dots \otimes \{a_m\})$  is the

depth  $m$  multiple Green function (Section 9 of [G1]):

$$G_\mu(a_0, \dots, a_m) := (2\pi i)^{-(2m-1)} \sum_T \text{sgn}(E_1 \wedge \dots \wedge E_{2m-1}) \int_{X^{m-1}} \omega_{2m-2}(G_{E_1} \wedge \dots \wedge G_{E_{2m-1}}). \quad (183)$$

The sum is over all plane trivalent trees  $T$  decorated by  $a_0, \dots, a_m$ . The integral is over the product of copies of  $X$  parametrised by the internal vertices of  $T$ . The function  $G_\mu(a_0, \dots, a_m)$  enjoys

- The dihedral symmetry:  $G_\mu(a_0, \dots, a_m) = G_\mu(a_1, \dots, a_m, a_0) = (-1)^{m+1} G_\mu(a_m, \dots, a_0)$ .
- The shuffle relations:  $\sum_{\sigma \in \Sigma_{p,q}} G_\mu(a_0, a_{\sigma(1)}, \dots, a_{\sigma(m)}) = 0$ .

The first property is clear from the definition. The second follows from Proposition 2.7.

**2. The Bloch-Wigner function.** The simplest multiple Green function appears when  $X = \mathbb{P}^1$ ,  $\mu = \delta_a$  and  $m = 3$ . It is described by a single Feynman diagram, shown on the left of Fig 32.

Recall the dilogarithm function  $\text{Li}_2(z)$ , its single valued version, the Bloch-Wigner function  $\mathcal{L}_2(z)$ , and the the cross-ratio  $r(z_1, \dots, z_4)$  of four points  $z_1, \dots, z_4$  on  $\mathbb{P}^1$  normalized by  $r(\infty, 0, 1, z) = z$ , see Section 1.3.

**Lemma 10.1** *One has*

$$G_a(a_0, a_1, a_2) = -\frac{1}{(2\pi i)^2} \mathcal{L}_2(r(a, a_0, a_1, a_2)).$$

**Proof.** We may assume without loss of generality that  $a = \infty$ . Then, up to an additive constant,  $G_\infty(x, y) = \log |x - y|$ . Therefore

$$G_\infty(a_0, a_1, a_2) = \frac{1}{(2\pi i)^3} \int_{\mathbb{CP}^1} \omega_2(\log |x - a_0| \wedge \log |x - a_1| \wedge \log |x - a_2|).$$

It remains to use the differential equations for the Hodge correlator and  $\mathcal{L}_2(z)$ .

**3. The classical polylogarithms.** Let  $X = \mathbb{CP}^1$  and  $s_0 = \infty$ . Consider the following cyclic word of length  $n + 1$ :

$$W_n = \mathcal{C}(\{1\} \otimes \{z\} \otimes \{0\} \otimes \dots \otimes \{0\}). \quad (184)$$

A  $W_n$ -decorated Feynman diagram with an internal vertex incident to two  $\{0\}$ -decorated external vertices evidently contributes the zero integral. There is a unique  $W_n$ -decorated Feynman diagram with no internal vertices incident to two  $\{0\}$ 's, see Fig 23. Denote by  $\mathbf{L}_n(z)$  the Hodge correlator for  $W_n$ , which uses the clockwise orientation of the tree.

Recall that the classical  $n$ -logarithm is a multivalued function on  $\mathbb{CP}^1 - \{0, 1, \infty\}$ , defined by an iterated integral:

$$Li_n(z) := \int_{0 \leq x_1 \leq \dots \leq x_n \leq z} \frac{dx_1}{1 - x_1} \wedge \frac{dx_2}{x_2} \wedge \dots \wedge \frac{dx_n}{x_n}.$$

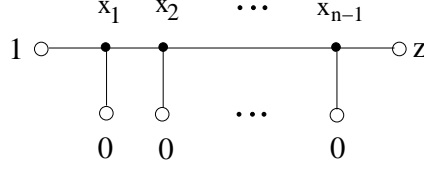


Figure 23: The Feynman diagram for the classical  $n$ -logarithm.

Here we integrate along a simplex consisting of points  $(x_1, \dots, x_n)$  on a path  $\gamma$  connecting 0 and  $z$ , following each other on the path. It has a single-valued version ([Z1], [BD]):

$$\mathcal{L}_n(z) := \begin{array}{l} \text{Re} \quad (n : \text{odd}) \\ \text{Im} \quad (n : \text{even}) \end{array} \left( \sum_{k=0}^{n-1} \beta_k \log^k |z| \cdot Li_{n-k}(z) \right), \quad n \geq 2.$$

Here  $\frac{2x}{e^{2x}-1} = \sum_{k=0}^{\infty} \beta_k x^k$ , so  $\beta_k = \frac{2^k B_k}{k!}$  where the  $B_k$  are the Bernoulli numbers. For example  $\mathcal{L}_2(z)$  is the Bloch - Wigner function.

Consider a modification  $\mathbb{L}_n^*(z)$  of the function  $\mathcal{L}_n(z)$  studied by A. Levin in [L]:

$$\mathbb{L}_n^*(z) := 4^{-(n-1)} \sum_{k \text{ even}; 0 \leq k \leq n-2} \binom{2n-k-3}{n-1} \frac{2^{k+1}}{(k+1)!} \mathcal{L}_{n-k}(z) \log^k |z|.$$

It is also handy to have a slight modification of this function:

$$\mathbb{L}_n(z) = 4^{(n-1)} \binom{2n-2}{n-1}^{-1} \mathbb{L}_n^*(z).$$

So  $\mathbb{L}_n(z) = \mathcal{L}_n(z)$  if and only if  $n \leq 3$ .

**Lemma 10.2**

$$-\mathbf{L}_n(z) = (2\pi i)^{-n} \mathbb{L}_n(z).$$

**Proof.** A direct integration carried out in Proposition 4.4.1 of [L] tells

$$-(2\pi i)^{-n} \mathbb{L}_n^*(z) = (2\pi i)^{-(2n-1)} \int_{(\mathbb{CP}^1)^{n-1}} \log |1-x_1| \bigwedge_{i=1}^{n-2} \left( d \log |x_i| \wedge d \log |x_i - x_{i+1}| \right) \wedge d \log |x_{n-1}| \wedge d \log |x_{n-1} - z|. \quad (185)$$

Proposition 6.2 in [GZ]<sup>8</sup> tells that given functions  $\varphi_i$  on a complex  $(n-1)$ -dimensional manifold  $M$ ,

$$\int_M \omega_{2n-2}(\varphi_0 \wedge \dots \wedge \varphi_{2n-2}) = (-4)^{n-1} \binom{2n-2}{n-1}^{-1} \int_M \varphi_0 d\varphi_1 \wedge \dots \wedge d\varphi_{2n-2}.$$

Using  $\varphi_i = \log |f_i|$  we see that integral (185) coincides up to a factor with the Hodge correlator integral for the counterclockwise orientation of the tree on Fig 23. The factor  $(-1)^{n-1}$  tells

<sup>8</sup>notice that it uses the form  $-\omega_m$  instead of  $\omega_m$

the difference between the counterclockwise and clockwise orientations of the tree. The lemma follows.

Alternatively, one can check using the differential equations for the function  $\mathcal{L}_n(z)$  that the function  $\mathbb{L}_n^*(z)$  satisfies the differential equation

$$d\mathbb{L}_n^*(z) = \frac{2n-3}{2n-2} \mathbb{L}_{n-1}^*(z) d^{\mathbb{C}} \log |z| - \frac{1}{2n-2} \log |z| d^{\mathbb{C}} \mathbb{L}_{n-1}^*(z).$$

The function  $\mathbf{L}_n(z)$  is a Hodge correlator, and thus satisfies the same differential equation (Section 6), and is zero if  $z = 0$ . This implies Proposition 10.2 by induction, since  $\mathbb{L}_n^*(z) = 0$ .

So the Hodge correlator  $\text{Cor}_{\mathcal{H}}$  delivers a single-valued version of the classical polylogarithm, while multiplying it by  $4^{-(n-1)} \binom{2n-2}{n-1}$ , i.e. using the normalized Hodge correlator  $\text{Cor}_{\mathcal{H}}^*$  (Section 6), we make the differential equation nicer.

**4. Cyclic multiple polylogarithms and Hodge correlators.** Let us set  $s_0 = \infty$ , and

$$\mathbb{L}_{k_0; \dots; k_m}(a_0 : \dots : a_m) := \text{Cor}_{\mathcal{H}, \infty}(\{a_0\} \otimes \{0\}^{\otimes k_0} \otimes \dots \otimes \{a_m\} \otimes \{0\}^{\otimes k_m}), \quad a_i \in \mathbb{C}^*. \quad (186)$$

We package these numbers into a generating series

$$\mathbb{L}(a_0 : \dots : a_m | u_0 : \dots : u_m) := \sum_{k_i=0}^{\infty} \mathbb{L}_{k_1; \dots; k_m}(a_0 : \dots : a_m) \frac{u_0^{k_0}}{k_0!} \dots \frac{u_m^{k_m}}{k_m!}.$$

The Hodge correlator integral assigned to the right diagram on Fig 24 equals  $\mathbf{L}_n(a_1/a_0)$ .

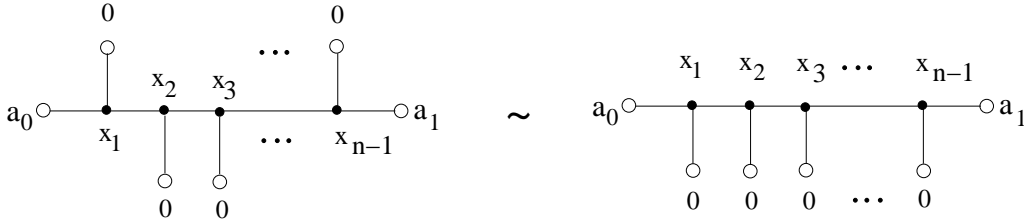


Figure 24: Feynman diagrams for the depth one Hodge correlators.

**Lemma 10.3** *One has*

$$\mathbb{L}(a_0 : a_1 | u_0 : u_1) = \mathbf{L}\left(\frac{a_1}{a_0} | u_1 - u_0\right). \quad (187)$$

**Proof.** The left hand side is a sum of the integrals attached to the diagrams on the left of Fig 24. The integral attached to the left Feynman diagram on Fig 24 coincides up to a sign, by its very definition, with the integral for the right Feynman diagram on Fig 24. The sign is

$$(-1)^{\text{the number of legs on the left diagram looking up}}.$$

Equivalently, the exponent is the number of legs which we need to flip in order to transform one diagram to the other. Indeed, there is a natural bijection between the edges of these two graphs.

It does not respect their canonical orientations. The sign measures the difference: flipping an edge we change the sign of the canonical orientation of the tree. It follows that

$$\begin{aligned} \mathbb{L}(a_0 : a_1 | u_0 : u_1) &\stackrel{\text{def}}{=} \sum_{k_0, k_1=0}^{\infty} \mathbb{L}_{k_0; k_1}(a_0 : a_1) \frac{u_0^{k_0}}{k_0!} \frac{u_1^{k_1}}{k_1!} = \\ \sum_{k_0, k_1=0}^{\infty} (-1)^{k_0} \mathbf{L}_{k_0+k_1}(a_0 : a_1) \frac{u_0^{k_0}}{k_0!} \frac{u_1^{k_1}}{k_1!} &= \sum_{k=0}^{\infty} \mathbf{L}_k(a_0 : a_1) \frac{(u_1 - u_0)^k}{k!} = \mathbf{L}\left(\frac{a_1}{a_0} | u_1 - u_0\right). \end{aligned}$$

The second equality was explained above. The lemma is proved.

**Cyclic multiple polylogarithms.** We are going to define them, following Section 8 of [G1], as a yet another generating series

$$\mathbf{L}(a_0 : \dots : a_m | t_0, \dots, t_m), \quad a_i \in \mathbb{C}^*, \quad t_i \in H_1(\mathbb{C}^*, \mathbb{R}), \quad t_0 + \dots + t_m = 0.$$

Consider a plane trivalent tree  $T$  decorated by  $m+1$  pairs  $(a_0, t_0), \dots, (a_m, t_m)$ . We picture  $a_i$ 's outside, and  $t_i$ 's inside of the circle. Each oriented edge  $\vec{E}$  of the tree  $T$  provides an element

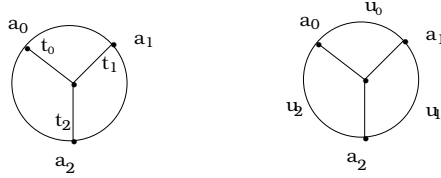


Figure 25: Decorations for the cyclic multiple polylogarithms.

$t(\vec{E})$  in  $H_1(\mathbb{C}^*, \mathbb{R})$  defined as follows. The edge  $E$  determines two trees rooted at  $E$ , see Fig 26. An orientation of the edge  $E$  allows to choose one of these trees: the one obtained by going in

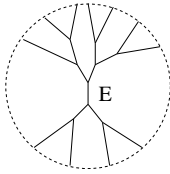


Figure 26: An edge  $E$  of a tree provides two trees rooted at  $E$ .

the direction shown by the orientation. The union of the incoming (i.e. different from  $E$ ) legs of these rooted trees coincides with the set of all legs of the initial tree. Let  $t(\vec{E})$  be the sum of all  $t_i$ 's corresponding to the incoming legs of one of these trees. The opposite orientation of the edge  $E$  produces the element  $-t(\vec{E})$ . We define the generating series for the cyclic multiple polylogarithms just like a Hodge correlator, with the generating series (187) for the classical polylogarithms serving as Green functions:

**Definition 10.4**

$$\mathbf{L}(a_0 : \dots : a_m | t_0, \dots, t_m) := (2\pi i)^{-(2m-1)} \sum_T \text{sgn}(E_1 \wedge \dots \wedge E_{2m-1}) \cdot \quad (188)$$

$$\int_{(\mathbb{CP}^1)^{m-1}} \omega_{2m-2} \left( \mathbf{L}(x_1^{E_1} : x_2^{E_1} | t_1^{E_1}, t_2^{E_1}) \wedge \dots \wedge \mathbf{L}(x_1^{E_{2m-1}} : x_2^{E_{2m-1}} | t_1^{E_{2m-1}}, t_2^{E_{2m-1}}) \right).$$

Here  $\mathbf{L}(x_1^E : x_2^E | t_1^E, t_2^E)$  is the function (187) assigned to the edge  $E$ , defined as follows. Choose an orientation of the edge  $E$ . It provides an ordered pair  $(x_1^E, x_2^E)$  of the vertices of  $E$ , as well as a pair of vectors  $(t_1^E, t_2^E) := (t(\vec{E}), -t(\vec{E}))$ . Thanks to (187) and the symmetry  $\mathbf{L}(a|t) = \mathbf{L}(a^{-1}|-t)$ , the function  $\mathbf{L}(x_1^E : x_2^E | t_1^E, t_2^E)$  does not change when we change the orientation of the edge  $E$ . The sum in (188) is over all plane 3-valent trees  $T$  with the given decoration. The integral is over product of  $\mathbb{CP}^1$ 's labeled by the internal vertices of  $T$ .

**Theorem 10.5**  $\mathbb{L}(a_0 : \dots : a_m | u_0 : \dots : u_m) = \mathbf{L}(a_0 : \dots : a_m | u_0 - u_m, u_1 - u_0, \dots, u_m - u_{m-1})$ .

**Proof.** *The depth 1 case.* This is Lemma 10.3. To proceed further, we need temporarily another generating series, whose combinatorics is illustrated on Fig 27:

$$\begin{aligned} & \mathbb{L}'(a_0 : \dots : a_m | s_0, t_0; \dots; s_m, t_m) := \\ & \sum_{n'_i, n''_i=0}^{\infty} \mathbb{L}_{n'_0, n''_0; \dots; n'_m, n''_m}(a_0 : \dots : a_m) \frac{s_0^{n'_0} t_0^{n''_0}}{n'_0! n''_0!} \dots \frac{s_m^{n'_m} t_m^{n''_m}}{n'_m! n''_m!}. \end{aligned} \quad (189)$$

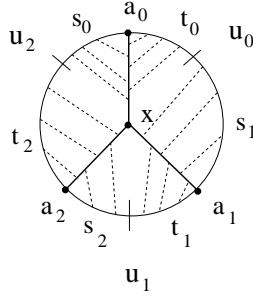


Figure 27: A Feynman diagram for the double polylogarithm.

*The depth 2 case.* The Hodge correlator integral is given by a Feynman diagram as on Fig 27. One has

$$\mathbb{L}'(a_0 : a_1 : a_2 | u_2, u_0; u_0, u_1; u_1, u_2) = \mathbb{L}(a_0 : a_1 : a_2 | u_0, u_1, u_2).$$

Indeed, to get the right hand side we have to substitute  $t_0 = s_1 = u_0$ ,  $t_1 = s_2 = u_1$ ,  $t_2 = s_0 = u_0$  into the sum (189), getting the left hand side. Furthermore, the same arguments about flipping the edges as in the proof of Lemma 10.3 show that

$$\mathbb{L}'(a_0 : a_1 : a_2 | s_0, t_0; s_1, t_1; s_2, t_2) = \mathbf{L}(a_0 : a_1 : a_2 | t_0 - s_0, t_1 - s_1, t_2 - s_2).$$

Here we integrate first over the internal vertices incident to the external  $\{0\}$ -decorated vertices. Thanks to Lemma 10.3 we get the  $\mathbf{L}$ -propagators assigned to the three “thick” edges of the tree. The remaining we get precisely the definition of the depth two function  $\mathbf{L}$ . Combining, we get

$$\mathbb{L}(a_0 : a_1 : a_2 | u_0, u_1, u_2) = \mathbf{L}(a_0 : a_1 : a_2 | u_0 - u_2, u_1 - u_0, u_2 - u_1).$$

*The general case.* It proceeds just like the depth 2 case. The case  $m = 3$  is illustrated on Fig 28. For example, we assign  $u_2 - u_0 = (u_2 - u_1) + (u_1 - u_0)$  to the internal edge  $E$  of the tree on Fig 28, in agreement with the recipe for  $t(\vec{E})$ . The theorem is proved.

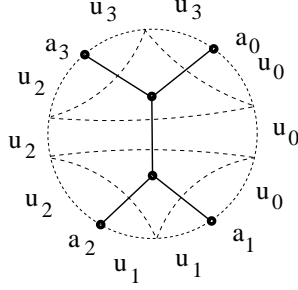


Figure 28: A Feynman diagram for the triple polylogarithm.

**5. Double shuffle relations.** We have defined the motivic cyclic multiple polylogarithms denoted below  $L_{n_0, \dots, n_m}^{\mathcal{M}}(a_0 : \dots : a_m)$ , in (179).

**Lemma 10.6** *i) The cyclic multiple polylogarithms enjoy the dihedral symmetry relations*

$$\begin{aligned} \mathbb{L}(a_0 : \dots : a_m | t_0 : \dots : t_m) &= \mathbb{L}(a_1 : \dots : a_m : a_0 | t_1 : \dots : t_m : t_0) = \\ &(-1)^{m+1} \mathbb{L}(a_m : \dots : a_0 | t_m : \dots : t_0). \end{aligned} \quad (190)$$

and the shuffle relations

$$\sum_{\sigma \in \Sigma_{p,q}} \mathbb{L}(a_0 : a_{\sigma(1)} \dots : a_{\sigma(m)} | t_0 : t_{\sigma(1)} : \dots : t_{\sigma(m)}) = 0. \quad (191)$$

*ii) The motivic cyclic multiple polylogs  $L^{\mathcal{M}}(a_0 : \dots : a_m | t_0 : \dots : t_m)$  satisfy the same relations.*

**Proof.** i) The first is clear from the definition. The second follows from Proposition 2.7. ii) is similar. The lemma is proved.

Set  $\mathbb{L}(a_0 : \dots : a_m) := \mathbb{L}_{0, \dots, 0}(a_0 : \dots : a_m)$  and

$$\mathbb{L}(a_0, \dots, a_m) := \mathbb{L}(1 : a_0, a_0 a_1, \dots, a_0 a_1 \dots a_{m-1}), \quad a_0 \dots a_m = 1.$$

We also need their motivic versions  $L^{\mathcal{M}}(a_0 : \dots : a_m)$  and  $L^{\mathcal{M}}(a_0, \dots, a_m)$ .

**Proposition 10.7** *i) For any  $p + q = m, p, q \geq 1$  one has the second shuffle relations*

$$\sum_{\sigma \in \Sigma_{p,q}} \mathbb{L}(a_0, a_{\sigma(1)}, \dots, a_{\sigma(m)}) = 0. \quad (192)$$

*ii) The same is true for the  $\mathcal{C}$ -motivic multiple logarithms  $L^{\mathcal{M}}(a_0, \dots, a_m)$ .*

**Proof.** ii) The elements  $L^{\mathcal{M}}(a_0 : \dots : a_m)$  satisfy the same coproduct formula as the generators of the dihedral Lie coalgebra, see (81) in [G4]:

$$\delta L^{\mathcal{M}}(a_0 : \dots : a_m) = \text{Cycle}_{m+1} \left( \sum_{k=1}^{m-1} L^{\mathcal{M}}(a_0 : \dots : a_k) \wedge L^{\mathcal{M}}(a_k : \dots : a_m) \right).$$



Indeed, this formula is equivalent to the one for the differential  $\delta_S$  in Section 5.2, which was made motivic in Section 9.

Let us prove the shuffle relations by the induction on  $m$ . Denote by  $s_2^{p,q}(a_0, \dots, a_m)$  the left hand side in (192). Theorem 4.3 in [G4] tells that these elements span a coideal. This implies by the induction that  $\delta s_2^{p,q}(a_0, \dots, a_m) = 0$ . Therefore  $s_2^{p,q}(a_0, \dots, a_m) \in \text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(m))$  in the category of variations over the base parametrising the  $a_i$ 's. Since  $m \geq 2$ ,  $\text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(m))$  is rigid, and thus is a constant. Setting  $a_i = 1$  in (191) we get  $L^{\mathcal{M}}(1 : \dots : 1) = 0$ . Thus  $s_2^{p,q}(1, \dots, 1) = 0$ , and hence the constant above is zero. The statement is proved.

i) Follows from ii) by applying the period map, or by using the differential equations.

So the cyclic multiple logarithms as well as their motivic avatars satisfy the double shuffle relations from Section 4 of [G4] on the nose, without lower depth corrections or products. The absence of the products is a general feature of the Lie-type elements / periods. The absence of the lower depth corrections is a remarkable fact. It implies that there is a homomorphism from the diagonal part of the dihedral Lie coalgebra of  $\mathbb{C}^*$  or  $F^*$  (*loc. cit.*) to the motivic Lie coalgebra which sends the standard generators of the former to the motivic cyclic multiple logarithms. Notice that the construction of a similar homomorphism in [G4] required the associate graded for the depth filtration of the motivic Lie coalgebra. Specifying to the roots of unity, we conclude that the mysterious connection between the geometry of modular varieties and motivic cyclic multiple logarithms at roots of unity is valid without going to the associated graded for the depth filtration.

So the generators of the dihedral Lie coalgebras are related to the cyclic multiple polylogarithms, rather than the usual multiple polylogarithms, which differ by the lower depth terms.

**Remark.** The double shuffle relations for the depth 2 motivic cyclic multiple polylogarithms follow from (190), since in this case they are equivalent to the dihedral symmetry relations. Their proof in general is more involved since the coproduct of the motivic cyclic multiple polylogarithms has the lower depth terms.

## 10.2 Hodge correlators on elliptic curves are multiple Eisenstein-Kronecker series

**1. The classical Eisenstein-Kronecker series.** Let  $E$  be a complex elliptic curve with a lattice of periods  $\Gamma$ , so that  $E = \mathbb{C}/\Gamma$ . The intersection form  $\Lambda^2\Gamma \rightarrow 2\pi i\mathbb{Z}$  leads to a pairing  $\chi : E \times \Gamma \rightarrow S^1$ . So a point  $a \in E$  provides a character  $\chi_a : \Gamma \rightarrow S^1$ . We denote by  $dz$  the differential on  $E$  provided by the coordinate  $z$  on  $\mathbb{C}$ . We normalize the lattice so that  $\Gamma = \mathbb{Z} \oplus \mathbb{Z}\tau$ ,  $\text{Im}\tau > 0$ .

Consider the generating series for the classical Eisenstein-Kronecker series

$$K(a|t) := \frac{\tau - \bar{\tau}}{2\pi i} \sum_{\gamma \in \Gamma - \{0\}} \frac{\chi_a(\gamma)}{|\gamma - t|^2}. \quad (193)$$

It depends on a point  $a$  of the elliptic curve and an element  $t$  in a formal neighborhood of zero in  $H_1(E, \mathbb{R})$ . It is invariant under the involution  $a \mapsto -a, t \mapsto -t$ . Expanding it into the power series in  $t$  and  $\bar{t}$  we get the classical Eisenstein-Kronecker series as the coefficients:

$$K(a|t) = \sum_{p,q \geq 1} \left( \frac{\tau - \bar{\tau}}{2\pi i} \sum_{\gamma \in \Gamma - \{0\}} \frac{\chi_a(\gamma)}{\gamma^p \bar{\gamma}^q} \right) t^{p-1} \bar{t}^{q-1}. \quad (194)$$

Set  $\text{Sym}_{p+q} F(x_1, \dots, x_{p+q}) := \sum_{\sigma} F(x_{\sigma(1)}, \dots, x_{\sigma(p+q)})$ . Consider the following cyclic word:

$$W_{p,q} := \mathcal{C}\left(\{0\} \otimes \{a\} \otimes \frac{d\bar{z}^p}{p!} \cdot \frac{dz^q}{q!}\right) := \text{Sym}_{p+q} \mathcal{C}\left(\{0\} \otimes \{a\} \otimes \frac{d\bar{z}^{\otimes p}}{p!} \otimes \frac{dz^{\otimes q}}{q!}\right). \quad (195)$$

The corresponding Hodge correlator is given by the Feynman diagram on Fig 29.

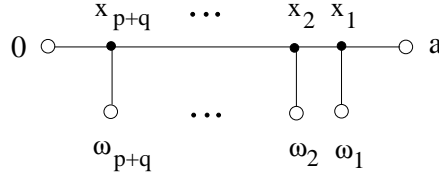


Figure 29: A Feynman diagram for the classical Eisenstein-Kronecker series;  $\omega_i = dz$  or  $d\bar{z}$ .

**Proposition 10.8** *i) The Hodge correlator of  $W_{p,q}$  is the classical Eisenstein-Kronecker series:*

$$\text{Cor}_{\mathcal{H}}(W_{p,q}) = \frac{(-1)^p}{2\pi i} \left( \frac{\tau - \bar{\tau}}{2\pi i} \right)^{p+q+1} \sum_{\gamma \neq 0} \frac{\chi_a(\gamma)}{\gamma^{p+1} \bar{\gamma}^{q+1}}.$$

*ii) The function  $K(a|t)$  coincides with the generating function of the Hodge correlators of (195).*

**Proof.** i) The normalized Green function  $g(x, y) = g(x - y)$  for the invariant volume form  $\text{vol}_E$  on  $E$  is a distribution defined as follows:

$$\chi_z(\gamma) := \exp\left(\frac{2\pi i(z\bar{\gamma} - \bar{z}\gamma)}{\tau - \bar{\tau}}\right), \quad g(z) := \frac{\tau - \bar{\tau}}{2\pi i} \sum_{\gamma \neq 0} \frac{\chi_z(\gamma)}{|\gamma|^2}, \quad (2\pi i)^{-1} \bar{\partial} \partial g(z) = \delta_0 - \text{vol}_E.$$

Therefore the Hodge correlator of (195) is obtained by integrating the following form over  $E^{p+q}$ :

$$\Omega_{p,q} := \frac{(-1)^p}{(2\pi i)^{(p+q+1)}} g(a - x_1) \bigwedge_{s=1}^p d\bar{x}_s \wedge \partial g(x_s - x_{s+1}) \wedge \bigwedge_{t=1}^q dx_{p+t} \wedge \bar{\partial} g(x_{p+t} - x_{p+t+1}).$$

Here  $x_{p+q+1} := 0$ . We used the clockwise orientation of the tree to make the product. Notice that the component of the form  $\omega_{p+q}$  responsible for this form comes with the coefficient  $(-1)^p p! q! / (p+q)!$ . The integral does not depend on the normalization of the Green function. Notice that

$$\partial \log g(x) = \sum_{\gamma \neq 0} \frac{\chi_a(\gamma)}{\gamma} dx, \quad \bar{\partial} g(x) = - \sum_{\gamma \neq 0} \frac{\chi_a(\gamma)}{\bar{\gamma}} d\bar{x}.$$

Thus we get

$$\int_{E^{p+q}} \Omega_{p,q} = \frac{(-1)^p}{2\pi i} \left( \frac{\tau - \bar{\tau}}{2\pi i} \right)^{p+q+1} \sum_{\gamma \neq 0} \frac{\chi_a(\gamma)}{\gamma^{p+1} \bar{\gamma}^{q+1}}.$$

Indeed, we have a convolution of the Green function with its first derivatives, which amounts to a product after the Fourier transform.

ii) Let  $l, \bar{l} \in H_1(E, \mathbb{C})$  be the homology classes dual to  $dz$  and  $d\bar{z}$ . The Hodge correlators of the elements (195) are naturally packaged into a generating function

$$\frac{(-1)^p}{2\pi i} \frac{\tau - \bar{\tau}}{2\pi i} \sum_{p,q=0}^{\infty} \left( \frac{\tau - \bar{\tau}}{2\pi i} \right)^{p+q} \sum_{\gamma \neq 0} \frac{\chi_a(\gamma)}{\gamma^{p+1} \bar{\gamma}^{q+1}} \bar{l}^p l^q. \quad (196)$$

This is naturally a function on a formal neighborhood of the origin in  $H_{DR}^1(E, \mathbb{C})(1)^+$ , where  $+$  means the complex conjugation acting on the forms on  $E$ . On the other hand  $K(a|t)$  is naturally a function on  $t \in \mathbb{C} = H_1(\Gamma) \otimes \mathbb{R} = H_1(E, \mathbb{R})$ . Indeed,  $\gamma$  and  $t$  in formula (193) lie in the same space!

The intersection pairing  $\langle *, * \rangle$  on  $H_1$  provides an isomorphism  $i : H_{DR}^1(E, \mathbb{C})(1)^+ \rightarrow H_1(E, \mathbb{R})$  such that  $\langle i(\omega), h \rangle := (\omega, h)$ . It follows that

$$\langle l, \bar{l} \rangle = \frac{2\pi i}{\tau - \bar{\tau}}, \quad i : dz \mapsto t := -\frac{\tau - \bar{\tau}}{2\pi i} \bar{l}, \quad i : d\bar{z} \mapsto \bar{t} := \frac{\tau - \bar{\tau}}{2\pi i} l. \quad (197)$$

Using this we get

$$(196) = \frac{1}{2\pi i} \frac{\tau - \bar{\tau}}{2\pi i} \sum_{\gamma \neq 0} \frac{\chi_a(\gamma)}{|\gamma - t|^2} = \frac{1}{2\pi i} K(a|t).$$

The Proposition is proved.

**Example.** It is easy to see that when  $a \in E$  is a torsion point, the coproduct of the averaged base point motivic correlator of  $\mathcal{C}(\{0\} \otimes \{a\} \otimes \text{Sym}^n \mathbb{H})$  (Section 9.5.4) is zero. So we get an element

$$\text{Cor}_{E-E_{\text{tors}}}^0(\mathcal{C}(\{0\} \otimes \{a\} \otimes \text{Sym}^n \mathbb{H})) \in \text{Ext}_{\mathcal{C}}^1(\mathbb{Q}(0), \text{Sym}^n \mathbb{H}(1)).$$

Its real period is given by the Eisenstien-Kronecker series. When  $E$  is a CM curve it gives the special values of the Hecke L-series with Grössencharacters. Summarizing, we get the motivic cohomology class corresponding to this special value. See [B], [Den1], [BL] for different approaches.

**2. Hodge correlators on an elliptic curve.** Let  $\mu$  be the unique invariant volume form on  $E$  of volume 1. The Green function corresponding to  $\mu$  is normalized by

$$\int_{E(\mathbb{C})} G_{\mu}(s, z) dz \wedge d\bar{z} = 0. \quad (198)$$

Below we consider the Hodge correlators decorated by the elements of

$$\mathcal{C}(\{a_0\} \otimes S^{k_0}(\Omega_E^1 \oplus \bar{\Omega}_E^1) \otimes \dots \otimes \{a_m\} \otimes S^{k_m}(\Omega_E^1 \oplus \bar{\Omega}_E^1)). \quad (199)$$

The symmetric Hodge correlators on an elliptic curve are linear combinations of those which appear when  $\{a_i\}$  are replaced by degree zero divisors on  $E$ . Abusing language, we call any Hodge correlator on an elliptic curve decorated by (199) and defined by using the normalized Green function  $G_{\mu}(x, y)$  a symmetric Hodge correlator. The *depth* of the Hodge correlator of  $W$  is the number of the  $S$ -factors of  $W$  minus one.

We package depth  $m$  symmetric Hodge correlators on  $E$  into non-holomorphic generating series:

$$K_{p_0, q_0; \dots; p_m, q_m}(a_0 : \dots : a_m) := \text{Cor}_{\mathcal{H}} \mathcal{C} \left( \{a_0\} \otimes \frac{dz^{p_0} d\bar{z}^{q_0}}{p_0! q_0!} \otimes \dots \otimes \{a_m\} \otimes \frac{dz^{p_m} d\bar{z}^{q_m}}{p_m! q_m!} \right).$$

$$K(a_0 : \dots : a_m | l_0 : \dots : l_m) = \sum_{p_i, q_i \geq 0} K_{p_0, q_0; \dots; p_m, q_m}(a_0 : \dots : a_m) \bar{l}_0^{p_0} l_0^{q_0} \dots \bar{l}_m^{p_m} l_m^{q_m}. \quad (200)$$

Following (197), we introduce the variables

$$t_i := -\frac{\tau - \bar{\tau}}{2\pi i} \bar{l}_i, \quad \bar{t}_i := \frac{\tau - \bar{\tau}}{2\pi i} l_i. \quad t_i, \bar{t}_i \in H_1(E, \mathbb{R}).$$

**3. The multiple Eisenstein-Kronecker series.** Our next goal is to present the symmetric Hodge correlators on elliptic curves as integrals of the generating series  $K(a|t)$ . We rewrite the function (194) as

$$\mathbf{K}(a_1 : a_2 | t_1, t_2) := K(a_1 - a_2 | t_1); \quad t_1 + t_2 = 0. \quad (201)$$

Let us define, following Section 8 of [G1], the *depth  $m$  multiple Eisenstein-Kronecker series*

$$\mathbf{K}(a_0 : \dots : a_m | t_0, \dots, t_m); \quad a_i \in E, \quad t_i \in H_1(E, \mathbb{R}), \quad t_0 + \dots + t_m = 0,$$

Consider a plane trivalent tree  $T$ , see Fig 25, decorated by  $m+1$  pairs  $(a_0, t_0), \dots, (a_m, t_m)$ . Each oriented edge  $\vec{E}$  of the tree  $T$  provides an element  $t(\vec{E}) \in H_1(E, \mathbb{R})$  just like in Section 10.1. We define the multiple Eisenstein-Kronecker series just like the multiple Green functions, with the generating series (201) for the classical Eisenstein-Kronecker series serving as Green functions:

$$\mathbf{K}(a_1 : \dots : a_{m+1} | t_1, \dots, t_{m+1}) := (2\pi i)^{-(2m-1)} \sum_T \text{sgn}(E_1 \wedge \dots \wedge E_{2m-1}) \cdot \quad (202)$$

$$\int_{E^{m-1}} \omega_{2m-2} \left( \mathbf{K}(x_1^{E_1} : x_2^{E_1} | t_1^{E_1}, t_2^{E_1}) \wedge \dots \wedge \mathbf{K}(x_1^{E_{2m-1}} : x_2^{E_{2m-1}} | t_1^{E_{2m-1}}, t_2^{E_{2m-1}}) \right).$$

Here  $\mathbf{K}(x_1^E : x_2^E | t_1^E, t_2^E)$  is the function (201) assigned to the edge  $E$ , defined just like in the rational case. The sum in (202) is over all plane 3-valent trees  $T$  cyclically labeled by the pairs  $(a_i, t_i)$ . The integral is over the product of copies of  $E$  labeled by the internal vertices of  $T$ .

Clearly

$$\mathbf{K}(a + a_0 : \dots : a + a_m | t_0, \dots, t_m) = \mathbf{K}(a_0 : \dots : a_m | t_0, \dots, t_m).$$

The multiple Eisenstein-Kronecker series enjoy the dihedral symmetry relations (190) and the shuffle relations (191) It is proved just the same way as Lemma 10.6.

**Theorem 10.9** *The generalized Eisenstein-Kronecker series are the generating series for the symmetric Hodge correlators on the elliptic curve  $E$ :*

$$K(a_0 : \dots : a_m | u_0 : \dots : u_m) := \mathbf{K}(a_0 : \dots : a_m | u_m - u_0, u_0 - u_1, \dots : u_{m-1} - u_m). \quad (203)$$

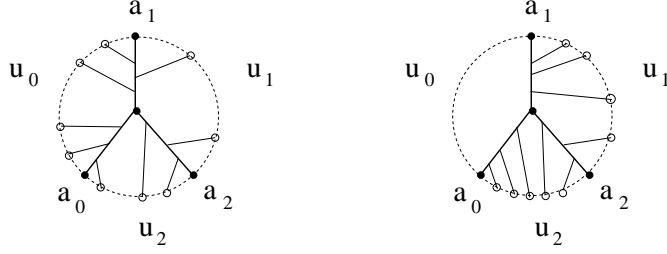


Figure 30: Feynman diagrams for symmetric Hodge correlators on an elliptic curve.

We visualize this as follows. The points  $a_i$  sit on a circle, and cut it into  $m + 1$  arcs. We assign the variables  $u_i$  to the arcs so that the vertex  $a_i$  shares the arcs labeled by  $u_{i-1}$  and  $u_i$ , see Fig 30.

**Proof.** A Feynman diagram decorated by a vector in (199), and which has an internal vertex incident to two  $\omega$ -decorated external vertices, contributes zero. Indeed, the correlator integral involves  $\int_E (\omega_1 \wedge \omega_2 + \omega_2 \wedge \omega_1) = 0$ . It follows that the only Feynman diagrams contributing to the Hodge correlators we consider are as on Fig 30.

Symmetric Hodge correlators of depths  $-1$  and  $0$  are zero. Indeed, every Feynman diagram with less than two  $S$ -decorated vertices has an internal vertex sharing two  $\omega$ -decorated edges.

The depth 1 case follows from Proposition 10.8. There are other depth 1 symmetric cyclic words, similar to the one on the left of Fig 24, where  $\{0\}$ 's are replaced by  $\omega_i$ 's. The integral for each of them coincides, up to a sign, with the one for the standard diagram on Fig 29. Namely, we just flip the  $\omega$ -decorated edges looking up, thus transforming the diagram to the one on Fig 29.

The rest of the proof repeats the proof of Theorem 10.5 with a slight modification stemming from the fact that  $H_1(E, \mathbb{R})$  is two-dimensional. The theorem is proved.

**Remark.** Let us show directly that the generating series (200) are invariant under the shift  $u_i \mapsto u_i - u_0$ . We use the same argument as in the proof of Lemma 10.3. Each external edge touching the arc labeled by  $u_0$  is flipped by moving its external vertex to the other arcs. For example, the right picture on Fig 30 represents the result of flipping of all external edges touching the  $u_0$ -arc on the left one. The correlator integral changes the sign under a flip of a single edge. Indeed, a flip changes only the orientation of the tree. This proves the claim.

**4. Polylogarithms on curves.** It is naturally to define *polylogarithms on curves* as the motivic/Hodge correlators of depth one. For the rational and elliptic curves we get this way the classical and elliptic polylogarithms. For the higher genus curves there is a yet smaller class of functions, defined in [G1] and discussed below. Their motivic avatars, however, do not span a Lie coalgebra.

Let  $X$  be a smooth compact complex curve of genus  $g \geq 1$ . and  $\mu$  is a volume one measure on  $X$ . Set

$$\mathbb{H}_{\mathbb{R}} := H_1(X, \mathbb{R}), \quad \mathbb{H}_{\mathbb{C}} := \mathbb{H}_{\mathbb{R}} \otimes \mathbb{C} = \mathbb{H}_{-1,0} \oplus \mathbb{H}_{0,-1}.$$

For each integer  $n \geq 1$  we define a 0-current  $G_n(x, y; \mu)$  on  $X \times X$  with values in

$$\mathrm{Sym}^{n-1} \mathbb{H}_{\mathbb{C}}(1) = \oplus_{s+t=n-1} S^p \mathbb{H}_{-1,0} \otimes S^q \mathbb{H}_{0,-1} \otimes \mathbb{R}(1). \quad (204)$$

The current  $G_1(x, y; \mu)$  is the Green function  $G_\mu(x, y)$ , The current  $G_n(x, y; \mu)$  for  $n > 1$  is given by a function on  $X \times X$ . To get complex valued functions out of the vector function  $G_n(x, y; \mu)$  we proceed as follows. Let

$$\overline{\Omega}_p \in S^p \overline{\Omega}^1, \quad \Omega_q \in S^q \Omega^1.$$

Then  $\overline{\Omega}_p \otimes \Omega_q$  is an element of the dual to (204). Let us define the pairing  $\langle G_n(x, y; \mu), \overline{\Omega}_p \cdot \Omega_q \rangle$ . Let

$$\Omega_p = \omega_{\alpha_1} \cdot \dots \cdot \omega_{\alpha_p}, \quad \Omega_q = \omega_{\beta_1} \cdot \dots \cdot \omega_{\beta_q}; \quad \omega_* \in \Omega^1.$$

Denote by  $p_i : X^{n-1} \rightarrow X$  the projection on  $i$ -th factor.

**Definition 10.10** *The  $n$ -logarithm on the curve  $X$  is given by*

$$\langle G_n(x, y; a), \overline{\Omega}_p \cdot \Omega_q \rangle :=$$

$$\frac{1}{(2\pi i)^n} \text{Alt}_{\{t_1, \dots, t_{n-1}\}} \left( \int_{X^{n-1}} \omega_{n-1} \left( G(x, t_1) \wedge G(t_1, t_2) \wedge \dots \wedge G(t_{n-1}, y) \right) \wedge \bigwedge_{i=1}^p p_i^* \overline{\omega}_{\alpha_i} \wedge \bigwedge_{j=1}^q p_{p+j}^* \omega_{\beta_j} \right).$$

Here we skewsymmetrize the integrand with respect to  $t_1, \dots, t_{n-1}$ . Before the skewsymmetrization the integrand depends on the element  $\overline{\omega}_{\alpha_1} \otimes \dots \otimes \overline{\omega}_{\alpha_p}$  from  $\otimes^p \overline{\Omega}^1$ ; after it depends only on its image in  $S^p \overline{\Omega}^1$ ; similarly for the second factor.

It is the Hodge correlator for the Feynman diagram on Fig 29. For example, the dilogarithm on a curve  $X$  is described by the second Feynman diagram on Fig 32. It is given by the integral

$$\langle G_2(x, y; \mu), \overline{\omega}_\alpha \rangle := \frac{1}{(2\pi i)^2} \int_X \omega_1 \left( G_\mu(x, t) \wedge G_\mu(t, y) \right) \wedge \overline{\omega}_\alpha(t).$$

**Remark.** Unlike in the genus  $\leq 1$  case, there are non-trivial Hodge correlators of depths  $-1$  and  $0$ . The simplest depth  $-1$  Hodge correlator is illustrated on Fig 31. Furthermore, there are other depth one Hodge correlators then the polylogarithms defined above.

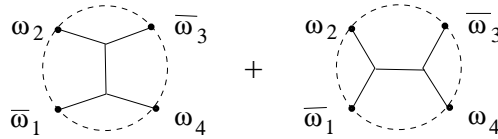


Figure 31: The simplest depth  $-1$  Hodge correlator.

Just like in the rational and elliptic cases, one can introduce cyclic multiple polylogarithms by taking the generating function of the polylogarithms on curves as the propagator assigned to an edge of a trivalent tree, and repeating the construction of Definition 10.4.

One proves just the same way as in Theorem 10.5, that these way we get all Hodge correlators corresponding the elements of  $\mathcal{C} \left( \{a_0\} \otimes S^{k_0}(\Omega_X^1 \oplus \overline{\Omega}_X^1) \otimes \dots \otimes \{a_m\} \otimes S^{k_m}(\Omega_X^1 \oplus \overline{\Omega}_X^1) \right)$ .

However, unlike in the cases of rational or elliptic curves, we do not get all Hodge correlators on curves this way, and the corresponding motivic correlators are not closed under the coproduct.

## 11 Motivic correlators on modular curves

In the Section  $X$  is a modular curve, and  $S$  is the set of cusps.

### 11.1 Hodge correlators on modular curves and generalized Rankin-Selberg integrals

**Rankin-Selberg integrals as Hodge correlators on modular curves.** For an arbitrary curve  $X$ , there are three different types of the Hodge correlators for the length three cyclic words. They correspond to the three Feynman diagrams on Fig. 32. The first is the double

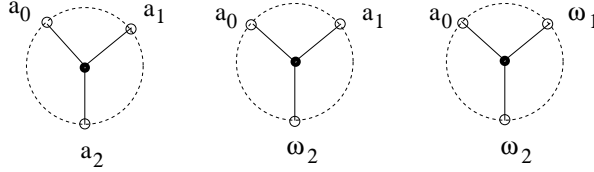


Figure 32: The three possible Feynman diagrams for the  $m = 2$  correlators.

Green function from Section 10.1. Let us interpret the other two in the case when  $X$  is a modular curve, and  $S$  is the set of its cusps. Let  $\omega_i$  be cuspidal weight two Hecke eigenforms. They are holomorphic 1-forms on  $X$ . Let  $s \in S$  be a cuspidal divisor on the modular curve  $X$ . Then  $G(s, t)$  is an Eisenstein series. By the Manin-Drinfeld theorem if  $\deg(s) = 0$ , there exists a rational function  $g_s$  on  $X$  such that  $\text{div} g_s = N \cdot s$ . So  $G(s, t) = N \log |g_s|^2$ . Setting  $a \sim_{\mathbb{Q}^*} b$  if  $a/b \in \mathbb{Q}^*$ , we get

$$\begin{aligned} \text{Cor}_{\mathcal{H}}\left(\left(\{a_0\} \otimes \{a_1\} \otimes \omega\right)\right) &= \int_{X(\mathbb{C})} G(a_0, t) d^{\mathbb{C}} G(a_1, t) \wedge \omega \sim_{\mathbb{Q}^*} \int_{X(\mathbb{C})} \log |g_{a_0}| d^{\mathbb{C}} \log |g_{a_1}| \wedge \omega. \\ \text{Cor}_{\mathcal{H}}\left(\mathcal{C}(\{a_0\} \otimes \omega_1 \otimes \overline{\omega}_2)\right) &= \int_{X(\mathbb{C})} G(a_0, t) \omega_1 \wedge \overline{\omega}_2 \sim_{\mathbb{Q}^*} \int_{X(\mathbb{C})} \log |g_a| \omega \wedge \overline{\omega}. \end{aligned}$$

Let  $L(\omega, s)$  be the  $L$ -function of the rank two weight  $-1$  motive  $M_{\omega}$  corresponding to the Hecke eigenform  $\omega$ , and  $L(\omega_1 \otimes \omega_2, s)$  the  $L$ -function of the product of motives corresponding to different Hecke eigenforms  $\omega_1$  and  $\omega_2$ . According to the classical Rankin-Selberg method the first integral is proportional, up to some standard factors, to  $L(\omega, 2)$ , and the second one to  $L(\omega_1 \otimes \omega_2, 2)$ . Moreover, there exist degree zero divisors  $a_0, a_1$  such that the proportionally coefficient is not zero [B].

**Generalized Rankin-Selberg integrals.** Let  $X$  be a modular curve, given together with its uniformization  $p_X : \mathcal{H} \rightarrow X$  by the hyperbolic plane  $\mathcal{H}$ . Choose a hyperbolic metric of curvature  $-1$  on  $X$ . Denote by  $\mu_X$  the corresponding volume 1 volume form on  $X$ . So  $p_X^* \mu_X$  is the standard volume form on the hyperbolic plane. Therefore if  $\pi : Y \rightarrow X$  is a natural map of modular curves, i.e. a map commuting with the uniformization maps, then the volume forms  $\mu_X$  and  $\mu_Y$  are compatible:  $\pi_* \mu_Y = \mu_X$ .

**Lemma 11.1** *Let  $\pi : Y \rightarrow X$  be a map of curves and  $\pi_*(\mu_Y) = \mu_X$  for certain measures  $\mu_Y, \mu_X$  on them. Let  $\pi_1 : Y \times Y \rightarrow X \times Y$  and  $\pi_2 : X \times Y \rightarrow X \times X$  be two natural projections. Then*

$$\pi_{1*} G_{\mu_Y}(y_1, y_2) = \pi_{2*} G_{\mu_X}(x_1, x_2) + C.$$

**Proof.** The Green function is defined uniquely up to adding a constant by the differential equation (56). The latter involves the volume form and the identity maps on the cohomology realized by the delta-currents  $\delta_\Delta$  and the Casimir elements for the space of holomorphic/antiholomorphic 1-forms. Clearly  $\pi_{1*}\delta_{\Delta_Y} = \pi_2^*\delta_{\Delta_X}$ . This implies a similar identity for the Casimirs. Finally,

$$\pi_{1*}(1_Y \otimes \mu_Y) = \deg \pi (1_X \otimes \mu_Y) = \pi_2^*(1_X \otimes \mu_X); \quad \pi_{1*}(\mu_Y \otimes 1_Y) = \mu_X \otimes 1_Y = \pi_2^*(\mu_X \otimes 1_X);$$

The lemma follows.

We say that Green functions  $G_{\mu_Y}$  and  $G_{\mu_X}$  are compatible if the constant in Lemma 11.1 is zero.

Choose a compatible family of Green functions  $G_{\mu_X}$  for the hyperbolic metrics on the modular curves  $X$ . We get a *hyperbolic Hodge correlator map* on the modular curves:

$$\text{Cor}_{\mathcal{H}, \mu_X}^h : \mathcal{CT}(V_{X,S}) \longrightarrow H^2(X).$$

The integral over the fundamental cycle of  $X$  provides an isomorphism  $H^2(X) \rightarrow \mathbb{C}$ . So we may assume that for an individual curve  $X$  the correlator is a complex number. However moving from  $X$  to  $Y$  we multiply it by  $\deg \pi$ .

**Proposition 11.2** *The hyperbolic Hodge correlator maps are compatible with natural projections  $\pi : Y \rightarrow X$  of modular curves:*

$$\text{Cor}_{\mathcal{H}, \mu_Y}^h \mathcal{C}(\pi^*v_1 \otimes \dots \otimes \pi^*v_m) = \pi^* \text{Cor}_{\mathcal{H}, \mu_X}^h \mathcal{C}(v_1 \otimes \dots \otimes v_m), \quad v_i \in V_{X,S}. \quad (205)$$

**Proof.** Lemma 11.1 plus the projection formula implies that, given 1-forms  $\omega_i$  on  $X$ , we have

$$\int_Y \pi^*\omega_1 \wedge \pi^*\omega_2 \cdot G_{\mu_Y}(y_1, y_2) = \int_X \omega_1 \wedge \omega_2 \cdot \pi_{1*}G_{\mu_Y}(y_1, y_2) = \int_X \omega_1 \wedge \omega_2 \cdot \pi_2^*G_{\mu_Y}(x_1, x_2). \quad (206)$$

Here the integrals are over  $y_1$  and  $x_1$ . Similarly, using (205) we have

$$G_{\mu_Y}(\pi^*\{s\}, y_2) = \pi_2^*G_{\mu_X}(s, x_2). \quad (207)$$

Therefore we have an identity similar to (206) where one or two of the decorating forms  $\omega_i$ 's are replaced by the decorating divisors  $\pi^*\{s_i\}$  – we understood (206) as the Hodge correlator for a tree with one internal trivalent vertex.

Let us use the induction on  $m$ . We understood the correlators are numbers via the isomorphism  $H^2(X) \xrightarrow{\sim} \mathbb{C}$ . The  $m = 3$  case follows from (207), using

$$\int_Y \pi^*\eta_1 \wedge \pi^*\eta_2 \cdot \pi^*\varphi_3 = \deg \pi \int_X \eta_1 \wedge \eta_2 \cdot \varphi_3.$$

For the induction step, take a plane trivalent tree  $T$  decorated by  $\pi^*v_1 \otimes \dots \otimes \pi^*v_m$ . Take two external edges sharing an internal vertex  $w$  and decorated by  $\pi^*v_i$  and  $\pi^*v_{i+1}$ . Applying (206) to the integral over the curve  $Y$  corresponding to  $w$ , we reduce the claim to the case of a tree with  $m - 1$  external vertices. The Proposition is proved.



Let  $\mathcal{M}$  be the universal modular curve. It is defined over  $\overline{\mathbb{Q}}$ . Since Hecke operators split the motive  $H^1(\mathcal{M})$  into a direct sum of the cuspidal and Eisenstein parts,  $\mathrm{gr}^W H^1 \mathcal{M} \otimes \overline{\mathbb{Q}} = H^1 \mathcal{M} \otimes \overline{\mathbb{Q}}$ . So one has

$$\mathbb{H}_{\mathcal{M}}^{\vee} = H^1 \mathcal{M} \otimes \overline{\mathbb{Q}} = \bigoplus_M M^{\vee} \otimes V_M \bigoplus \mathbb{Q}(\widehat{\mathrm{Cusps}})(-1). \quad (208)$$

Here the first sum is over all pure rank 2, weight  $-1$  motives  $M$  over  $\mathbb{Q}$ . The  $\overline{\mathbb{Q}}$ -vector space  $V_M$  is a representation of the finite adele group  $GL_2(\mathbb{A}_{\mathbb{Q}}^f)$ . The tensor product of the Steinberg representation of  $GL_2(\mathbb{R})$  and  $V_M$  is an automorphic representation of  $GL_2(\mathbb{A}_{\mathbb{Q}})$  corresponding to the pure motive  $M$  via the Langlands correspondence.

In the de Rham realization, we get an adelic description of  $H_{DR}^1(\mathcal{M}, \mathbb{C})$ :

$$H_{DR}^1(\mathcal{M}, \mathbb{C}) = \bigoplus_M (\Omega_M^1 \oplus \overline{\Omega}_M^1) \otimes V_M \bigoplus \mathbb{Q}(\widehat{\mathrm{Cusps}})(-1).$$

Here  $\Omega_M^1$  is the  $(1, 0)$ -part of the de Rham realization of the motive  $M$ . Proposition 11.2 implies that the hyperbolic Hodge correlator map provides a  $GL_2(\mathbb{A}_{\mathbb{Q}}^f)$ -covariant map

$$\mathrm{Cor}_{\mathcal{H}}^h : \mathrm{CT}(H_{DR}^1(\mathcal{M}, \mathbb{C}))(1) \rightarrow \mathbb{C}.$$

Indeed, it is  $GL_2(\widehat{\mathbb{Z}})$ -covariant on the nose, and  $GL_2(\mathbb{Q})$ -covariant thanks to the compatibility with the projections established in Proposition 11.2. So it descends to a map of  $GL_2(\mathbb{A}_{\mathbb{Q}}^f)$ -coinvariants

$$\mathrm{Cor}_{\mathcal{H}}^h : \mathrm{CT}(H_{DR}^1(\mathcal{M}, \mathbb{C}))(1)_{GL_2(\mathbb{A}_{\mathbb{Q}}^f)} \longrightarrow \mathbb{C}. \quad (209)$$

The map (209) is a generalization of the Rankin-Selberg integrals: the latter appear for the triple tensor product of copies of  $H_{DR}^1(\mathcal{M}, \mathbb{C})$ , as explained in 11.1.1. Restricting this map to the subspace  $\overline{\mathcal{CLie}}^{\vee}(H_{DR}^1(\mathcal{M}, \mathbb{C}))$  of the cyclic tensor product of  $H_{DR}^1(\mathcal{M}, \mathbb{C})$  (Section 9.4) we get the periods of  $\pi_1^{\mathrm{nil}}(\mathcal{M})$ .

## 11.2 Motivic correlators in towers

Let  $\pi : X' \rightarrow X$  be a nontrivial map of projective curves,  $S'$  and  $S$  are subsets of points of  $X'$  and  $X$ , and  $\pi(S') = S$ . We say that vectors  $v_{s'}$  and  $v_s$  at the points  $s' \in S'$  and  $s \in S$  are compatible with respect to the map  $\pi$  if for local parameters  $t'$  and  $t$  such that  $dt'$  and  $dt$  are dual to the vectors  $v'$  and  $v$  the restriction of  $(t')^{m_{\pi}(s')}/\pi^*t$  to  $s'$  equals 1, where  $m_{\pi}(s')$  is the multiplicity of the map  $\pi$  at the point  $s'$ . In particular, if  $m_{\pi}(s') = 1$ , this means that  $d\pi_*(v') = v$ . Let us assume that for each  $s \in S$  (respectively  $s' \in S'$ ) there is a non-zero vector  $v_s \in T_s X$  (respectively  $v_{s'} \in T_{s'} X$ ), and that these vectors are compatible under the map  $\pi$ . Denote by  $\deg(\pi)$  the degree of the map  $\pi$ . Recall

$$\mathbb{H}_{X,S}^{\vee} := H^1(X) \oplus \mathbb{Q}[S](-1); \quad \mathbb{H}_{X,S^*}^{\vee} \cong \mathrm{gr}^W H^1(X - S).$$

There are maps

$$\pi^* : \mathbb{H}_{X,S}^{\vee} \longrightarrow \mathbb{H}_{X',S'}^{\vee}, \quad \pi_* : \mathbb{H}_{X',S'}^{\vee} \longrightarrow \mathbb{H}_{X,S}^{\vee}, \quad \pi_* \pi^* = \deg(\pi). \quad (210)$$

$$\pi_S^* : \mathbb{Q}[S] \rightarrow \mathbb{Q}[S']; \quad \{s\} \rightarrow \sum_{s' \in \pi^{-1}(s)} m_{\pi}(s') \{s'\}$$

So  $\deg \pi_S^*(\{s\}) = \deg \pi$ . The maps  $\pi^*$  and  $\pi_S^*$  induce a map of vector spaces

$$\pi_C^* : \mathcal{CT}(\mathbb{H}_{X,S}^\vee) \otimes \mathbb{Q}[S] \longrightarrow \mathcal{CT}(\mathbb{H}_{X',S'}^\vee) \otimes \mathbb{Q}[S'].$$

**Lemma 11.3** *Assume that the tangential base points at  $S'$  and  $S$  are compatible with the map  $\pi : X' - S' \rightarrow X - S$ . Then there is a commutative diagram, with the horizontal rows provided by (178), and the vertical maps are induced by the map  $\pi^*$ :*

$$\begin{array}{ccc} \mathcal{CT}(\mathbb{H}_{X',S'}^\vee) \otimes \mathbb{Q}[S'] & \xrightarrow{\text{Cor}_{X'-S'}} & \mathcal{L}_{\text{Mot}} \otimes H^2(X') \\ \uparrow \pi_C^* & & \uparrow \\ \mathcal{CT}(\mathbb{H}_{X,S}^\vee) \otimes \mathbb{Q}[S] & \xrightarrow{\text{Cor}_{X-S}} & \mathcal{L}_{\text{Mot}} \otimes H^2(X) \end{array}$$

**Proof.** If  $\pi$  is unramified at every point of  $S'$  it follows straight from the definition. The general case is deduced from this, say, by the specialization. The lemma is proved.

Let  $\text{tr}_X : H^2(X) \rightarrow \mathbb{Q}(-1)$  be the canonical map. Since  $\text{tr}_{X'} \pi^* = \deg(\pi) \text{tr}_X$ , we get a commutative diagram

$$\begin{array}{ccc} \mathcal{CT}(\mathbb{H}_{X',S'}^\vee) \otimes \mathbb{Q}[S'](1) & \xrightarrow{\text{Cor}_{X'-S'}} & \mathcal{L}_{\text{Mot}} \\ \uparrow \pi^* \otimes \deg(\pi)^{-1} \pi_C^* & & \uparrow = \end{array} \quad (211)$$

$$\mathcal{CT}(\mathbb{H}_{X,S}^\vee) \otimes \mathbb{Q}[S](1) \xrightarrow{\text{Cor}_{X-S}} \mathcal{L}_{\text{Mot}}$$

Now suppose that we have a projective system  $(X_\alpha, S_\alpha)$  of curves  $X_\alpha$  with finite subsets of points  $S_\alpha$  on them and maps between them as above. Using the normalized maps  $\deg(\pi)^{-1} \pi_C^*$ , functions on the sets  $S_\alpha$  in the limit turn into measures on the projective limit  $\widehat{S} := \lim S_\alpha$ . We denote the space of measures by  $\text{Meas}(\widehat{S})$ . So using (211), we get a map of the corresponding inductive limit

$$\lim_{\longrightarrow} \mathcal{CT}(\mathbb{H}_{X_\alpha, S_\alpha}^\vee) \otimes \text{Meas}(\widehat{S})(1) \longrightarrow \mathcal{L}_{\text{Mot}}. \quad (212)$$

Here  $\mathcal{X}$  denotes the projective limit of the curves  $X_\alpha - S_\alpha$ . Abusing notation, we denote below the inductive limit in (212) by  $\mathcal{CT}(\mathbb{H}_{\mathcal{X}}^\vee) \otimes \text{Meas}(\widehat{S})(1)$ .

**Examples.** 1. Let  $E$  be an elliptic curve. The curves  $E - E[N]$  and the isogenies  $E - E[MN] \rightarrow E - E[N]$  form a tower. We assign to every missing point on  $E - E[N]$  the tangent vector  $\frac{1}{N}v_\Delta$ . We get a compatible family of tangential base points for the tower  $\{E - E[N]\}$ . Therefore we arrive at the motivic correlator map

$$\text{Cor}_{\mathcal{E}} : \left( \mathcal{CT}(\mathbb{H}_{\mathcal{E}}^\vee) \otimes \text{Meas}(\widehat{E_{\text{tors}}})(1) \right)_{\widehat{E_{\text{tors}}}} \longrightarrow \mathcal{L}_{\text{Mot}}.$$

There is a unique  $\widehat{E_{\text{tors}}}$ -invariant measure  $\mu_{\mathcal{E}}^0$  on the Tate module  $\widehat{E_{\text{tors}}}$ . So we get a map

$$\text{Cor}_{\mathcal{E}}^0 : \mathcal{CT}(\mathbb{H}_{\mathcal{E}}^\vee)(1) \otimes \mu_{\mathcal{E}}^0 \longrightarrow \mathcal{L}_{\text{Mot}}.$$

2. Similarly, when  $X = P^1$  and  $S = \{0\} \cup \{\infty\} \cup \mu_{l^\infty}$  we get the measures on  $\mathbb{Z}_l(1)$ .

**Complements.** Abusing notation, we call by  $\pi_C^*$  the left vertical map in (211). It is not a map of Lie coalgebras. However it will become a Lie coalgebra map if we consider the following quotient of the Lie coalgebra  $\mathcal{CT}(\mathbb{H}_{X',S'}^\vee)(1)$ :

**Definition 11.4**  $\mathcal{C}'T(\mathbb{H}_{X',S'}^\vee)$  is the quotient of  $\mathcal{CT}(\mathbb{H}_{X',S'}^\vee)$  by the subobject spanned by expressions

$$\mathcal{C}\left(\pi^*v_1 \otimes \dots \otimes \pi^*v_{m-1} \otimes v_m\right), \quad v_1, \dots, v_{m-1} \in \mathbb{H}_{X,S}^\vee, \quad v_m \in \text{Ker}\pi_* \subset \mathbb{H}_{X',S'}^\vee. \quad (213)$$

**Lemma 11.5** The correlator map  $\text{Cor}_{X-S}$  annihilates elements

$$\mathcal{C}\left(\pi^*v_1 \otimes \dots \otimes \pi^*v_{m-1} \otimes v_m\right) \otimes \pi_S^*\{s\}(1), \quad \pi_*v_m = 0.$$

**Proof.** We prove this in the  $l$ -adic realization. A proof for another realizations is similar. It is sufficient to prove the case when  $\pi : X' - S' \rightarrow X - S$  is a Galois cover with a group  $G$ . The action of  $\text{Gal}(\overline{F}/F)$  commutes with the action of  $G$ . Thus

$$\begin{aligned} \text{Cor}_{X'-S'}\mathcal{C}(\pi^*v_1 \otimes \dots \otimes \pi^*v_{m-1} \otimes v_m) \otimes \pi_S^*\{s\} &= \text{Cor}_{X'-S'} \frac{1}{|G|} \sum_{g \in G} g^* \mathcal{C}(\pi^*v_1 \otimes \dots \otimes \pi^*v_{m-1} \otimes v_m) \otimes \pi_S^*\{s\} = \\ &= \text{Cor}_{X'-S'}\mathcal{C}(\pi^*v_1 \otimes \dots \otimes \pi^*v_{m-1} \otimes \pi_*v_m) \otimes \pi_S^*\{s\} = 0. \end{aligned}$$

The lemma is proved.

Here is an analytic counterpart:

**Lemma 11.6** Let  $\pi : X' \rightarrow X$  and  $\pi_*(\mu_{X'}) = \mu_X$ . Then

$$\text{Cor}_{\mathcal{H}, \mu_{X'}}\mathcal{C}\left(\pi^*v_1 \otimes \dots \otimes \pi^*v_{m-1} \otimes v_m\right) = 0, \quad m \geq 3, \quad \pi_*v_m = 0.$$

**Proof.** We prove it by the induction on  $m$ . The  $m = 3$  case is clear from (206). Take a plane trivalent tree  $T$  decorated by  $\pi^*v_1 \otimes \dots \otimes \pi^*v_{m-1} \otimes v_m$ . Take two external edges sharing an internal vertex  $w$  and decorated by  $\pi^*v_i$  and  $\pi^*v_{i+1}$ . Using (206) for the integral over the curve assigned to  $w$  we reduce the claim to the case of a tree with  $m - 1$  external vertices. The lemma is proved.

**Lemma 11.7**  $\mathcal{C}'T(\mathbb{H}_{X',S'}^\vee)(1)$  inherits a Lie coalgebra structure;  $\pi_C^*$  is a Lie coalgebra map.

**Proof.** Let us check that elements (213) span a coideal for the cobracket  $\delta = \delta_{\text{Cas}} + \delta_S$ . A typical term of  $\delta_{\text{Cas}}(213)$  is  $\sum_k \mathcal{C}(M_1 \otimes \alpha_k) \wedge \mathcal{C}(\alpha_k^\vee \otimes M_2)$ , where  $\sum_k \alpha_k \otimes \alpha_k^\vee$  is the Casimir element in  $H^1(X') \otimes H^1(X')$ , and  $\mathcal{C}(M_1 \otimes M_2)$  is the element (213). There is a decomposition

$$H^1(X') = \pi^*H^1(X) \oplus \pi^*H^1(X)^\perp.$$

The summands are orthogonal for the intersection pairing on  $H^1(X')$ . The map  $\pi_*$  annihilates the second summand. We may assume that  $v_m$  enters  $M_2$ . If  $\alpha_k \in \pi^*H^1(X)$ , then  $\alpha_k^\vee \in \pi^*H^1(X)$ , and  $\mathcal{C}(\alpha_k^\vee \otimes M_2)$  is of type (213). Otherwise  $\alpha_k, \alpha_k^\vee \in \pi^*H^1(X)^\perp$ , thus  $\mathcal{C}(\alpha_k \otimes M_1)$  is of type (213). The  $\delta_S$  term deals with the  $S$ -decorated vertices. Its typical term is

$$\sum_i \mathcal{C}(M_1 \otimes s_i) \wedge \mathcal{C}(s_i \otimes M_2) = \sum_i \mathcal{C}(M_1 \otimes (s_i - s_1)) \wedge \mathcal{C}(s_i \otimes M_2) + \mathcal{C}(M_1 \otimes s_1) \wedge \mathcal{C}\left(\sum_i s_i \otimes M_2\right).$$

The left factor of the first summand, and the right factor of the second are of type (213).

**Corollary 11.8** Set  $\mu_S := \frac{1}{|S|} \sum_{s \in S} \{s\}$ . The map  $\text{Cor}_{X-S}$  gives rise to a Lie coalgebra map.

$$\text{Cor}_{X-S}^{\text{av}} : \mathcal{C}'\text{T}(\mathbb{H}_{X,S}^{\vee}) \otimes \mu_S \otimes H_2(X) \longrightarrow \mathcal{L}_{\text{Mot}}.$$

**Proof.** Follows from Lemmas 11.5 and 11.7.

### 11.3 Motivic correlators for the universal modular curve

Recall the universal modular curve  $\mathcal{M}$  and decomposition (208). The tower of modular curves has a natural family of tangential base points at the cusps provided by exponents of the canonical parameter  $z$  on the upper half plane. Precisely, if the stabilizer of a cusp is an index  $N$  subgroup of the maximal unipotent subgroup in  $SL_2(\mathbb{Z})$ , it is  $\exp(2\pi iz/N)$ . They are compatible, in the sense of Section 11.2, with the natural projections of the modular curves. Going to the limit in the tower of modular curves, as explained in Section 11.2, we arrive at the following picture. There is a Lie coalgebra in the  $\otimes$ -category of pure motives generated by  $H^1(\overline{\mathcal{M}})$ :

$$\mathcal{C}'\text{T}(\mathbb{H}_{\mathcal{M}}^{\vee}) \otimes \text{Meas}(\widehat{\text{Cusps}})(1).$$

The group  $GL_2(\widehat{\mathbb{Z}})$  acts by its automorphisms. The group  $GL_2(\mathbb{Q})$  acts by its automorphisms thanks to the compatibility with projections (Section 11.2). So the group  $GL_2(\mathbb{A}_{\mathbb{Q}}^f)$  acts. Moreover, the canonical map of Lie coalgebras

$$\text{Cor}_{\mathcal{M}} : \mathcal{C}'\text{T}(\mathbb{H}_{\mathcal{M}}^{\vee}) \otimes \text{Meas}(\widehat{\text{Cusps}})(1) \longrightarrow \mathcal{L}_{\text{Mot}}$$

is  $GL_2(\mathbb{A}_{\mathbb{Q}}^f)$ -equivariant. (The group acts trivially on the right). So it descends to a map of the coinvariants

$$\text{Cor}_{\mathcal{M}} : \left( \mathcal{C}'\text{T}(\mathbb{H}_{\mathcal{M}}^{\vee}) \otimes \text{Meas}(\widehat{\text{Cusps}})(1) \right)_{GL_2(\mathbb{A}_{\mathbb{Q}}^f)} \longrightarrow \mathcal{L}_{\text{Mot}}.$$

In the de Rham realization there is a commutative diagram

$$\begin{array}{ccc} \left( \mathcal{C}'\text{T}(H_{DR}^1(\mathcal{M})) \otimes \text{Meas}(\widehat{\text{Cusps}})(1) \right)_{GL_2(\mathbb{A}_{\mathbb{Q}}^f)} & \xrightarrow{\text{Cor}_{\mathcal{M}}} & \mathcal{L}_{\text{Mot}} \\ & \text{Cor}_{\mathcal{H}} \searrow \downarrow \mathcal{P} & \\ & \mathbb{C} & \end{array}$$

where the diagonal arrow is the Hodge correlator map, and the vertical one is the period map.

**Lemma 11.9** There is a unique  $GL_2(\mathbb{A}_{\mathbb{Q}}^f)$ -covariant volume 1 measure  $\mu^0 \in \text{Meas}(\widehat{\text{Cusps}})$ ,

**Proof.** Let  $U$  be the upper triangular unipotent subgroup of  $GL_2$  and  $B$  the normalizing it Borel subgroup. Then there are isomorphisms

$$\widehat{\text{Cusps}} = GL_2(\widehat{\mathbb{Z}})/U(\widehat{\mathbb{Z}}) = GL_2(\mathbb{A}_{\mathbb{Q}}^f)/B(\mathbb{Q})U(\mathbb{A}_{\mathbb{Q}}^f).$$

The first is obvious, the second follows from the Iwasawa decomposition  $GL_2(\mathbb{A}_{\mathbb{Q}}^f) = B(\mathbb{Q})GL_2(\widehat{\mathbb{Z}})$ . A Haar measure on  $GL_2(\mathbb{A}_{\mathbb{Q}}^f)$  induces a covariant measure on the quotient. We normalize it by the volume one condition. The lemma is proved.

Taking the covariant measure  $\mu^0$  we are getting a map

$$\text{Cor}_{\mathcal{M}}^0 : \mathcal{C}'\text{T}(\mathbb{H}_{\mathcal{M}}^{\vee})(1)_{GL_2(\mathbb{A}_{\mathbb{Q}}^f)} \otimes \mu^0 \longrightarrow \mathcal{L}_{\text{Mot}}. \quad (214)$$

**Lemma 11.10** *The motivic correlators for the classical Rankin-Selberg integrals (Section 11.1) are motivic  $\text{Ext}^1$ 's:*

$$\text{Cor}_{\mathcal{M}}^0 \mathcal{C}(\{a_0\} \otimes \{a_1\} \otimes M_{\omega}^{\vee}) \in \text{Ext}_{\text{Mot}}^1(\mathbb{Q}(0), M_{\omega}(1)).$$

$$\text{Cor}_{\mathcal{M}}^0 \mathcal{C}(\{a_0\} \otimes M_{\omega_1}^{\vee} \otimes M_{\omega_2}^{\vee}) \in \text{Ext}_{\text{Mot}}^1(\mathbb{Q}(0), M_{\omega_1} \otimes M_{\omega_2}).$$

**Proof.** Let us compute the second coproduct. Set  $\mathbb{H} := \mathbb{H}_{\overline{\mathcal{M}}}$ .

$$\delta \text{Cor}_{\mathcal{M}}^0 \mathcal{C}(\{a_0\} \otimes M_{\omega_1}^{\vee} \otimes M_{\omega_2}^{\vee}) = \text{Cor}_{\mathcal{M}}^0 \mathcal{C}(\{a_0\} \otimes \mathbb{H}) \wedge \text{Cor}_{\mathcal{M}}^0 \mathcal{C}(\mathbb{H} \otimes M_{\omega_1}^{\vee} \otimes M_{\omega_2}^{\vee}).$$

One has  $\text{Cor}_{\mathcal{M}} \mathcal{C}(\{a_0\} \otimes \mathbb{H})(1) \otimes \{s\} = a_0 - s \in J_X \otimes \mathbb{Q} = \text{Ext}_{\text{Mot}}^1(\mathbb{Q}(0), \mathbb{H})$ . Since  $s, a_0$  are cusps, it is zero by the Manin-Drinfeld theorem. Thus  $\text{Cor}_{\mathcal{M}}^0 \mathcal{C}(\{a_0\} \otimes \mathbb{H})(1) = 0$ , and hence the coproduct is zero. This implies the second claim. The proof of the first claim is similar. The lemma is proved.

For certain divisors  $s_i$  these elements are not zero, and hypothetically generate the  $\text{Ext}^1$ -groups.

How to describe the image of the motivic correlator map (214)? Recall the hypothetical abelian category  $\mathcal{C}_{\mathcal{M}}$  of mixed motives generated by  $H^1(\mathcal{M})$ . Let  $\mathcal{P}_{\mathcal{M}}$  be the semi-simple category of pure motives generated by  $H^1(\mathcal{M})$ , The motivic Lie algebra of the mixed category  $\mathcal{C}_{\mathcal{M}}$  is supposed to be a free Lie algebra in the category  $\mathcal{P}_{\mathcal{M}}$ . Let  $\mathcal{L}_{\mathcal{M}}$  be the dual motivic Lie coalgebra. The map (214) lands in  $\mathcal{L}_{\mathcal{M}}$ :

$$\text{Cor}_{\mathcal{M}}^0 : \mathcal{C}'\text{T}(\mathbb{H}_{\mathcal{M}}^{\vee})(1)_{GL_2(\mathbb{A}_{\mathbb{Q}}^f)} \otimes \mu^0 \longrightarrow \mathcal{L}_{\mathcal{M}}. \quad (215)$$

The space of its cogenerators of type  $N^{\vee}$ , where  $N$  is a pure motive of negative weight from  $\mathcal{P}_{\mathcal{M}}$ , should be isomorphic to  $\text{Ext}^1(\mathbb{Q}(0), N) \otimes N^{\vee}$ ; Beilinson's conjectures predict its dimension.

The map (215) is not surjective for a very simple reason. The motivic correlators do not give non-trivial elements of the Jacobian  $J_{\mathcal{M}} \otimes \mathbb{Q}$  of the universal modular curve  $\mathcal{M}$ . Indeed, we get in the image of (215) only the images of degree zero divisors supported at the cusps (Section 9.4.1), which are zero by the Manin-Drinfeld theorem. On the other hand the simplest component of  $\mathcal{L}_{\mathcal{M}}$  is given by  $\text{Ext}^1(\mathbb{Q}(0), H_1(\mathcal{M})) = J_{\mathcal{M}} \otimes \mathbb{Q}$ . However Lemma 11.10 and the Rankin-Selberg method suggest that we do get  $\text{Ext}^1(\mathbb{Q}(0), N)$  for  $N = H_1(\mathcal{M})(1)$  and  $N = H_1(\mathcal{M}) \otimes H_1(\mathcal{M})$ .

More specifically, let  $V_{\omega_i}$  (resp.  $M_{\omega_i}$ ) be the representation of  $GL_2(\mathbb{A}_{\mathbb{Q}}^f)$  (resp. the weight  $-1$  motive) corresponding to a Hecke eigenform  $\omega_i$ . The motivic correlator provides a map

$$\text{Cor}_{\mathcal{M}}^0 : \mathcal{C}'\left(\bigotimes_{i=1}^k M_{\omega_i}^{\vee} \otimes V_{\omega_i}\right)(1) \longrightarrow \mathcal{L}_{\bigotimes_{i=1}^k M_{\omega_i}(-1)} \bigotimes_{i=1}^k M_{\omega_i}^{\vee}(1). \quad (216)$$

It is essentially described by a map of vector spaces  $\mathcal{C}'\left(\bigotimes_{i=1}^k V_{\omega_i}\right) \longrightarrow \mathcal{L}_{\bigotimes_{i=1}^k M_{\omega_i}(-1)}$ .

**Examples.** 1. Consider the motivic correlator corresponding to the tensor product of the motives of Hecke eigenforms  $\omega_1, \dots, \omega_4$  as on Fig 31. Find vectors  $v_{\omega_i} \in V_{\omega_i}$  providing non-zero elements

$$\mathrm{Cor}_{\mathcal{M}}^0 \mathcal{C} \left( \bigotimes_{i=1}^4 M_{\omega_i}^{\vee} \otimes v_{\omega_i} \right) (1) \in \mathrm{Ext}_{\mathcal{M}\mathcal{M}}^1 (\mathbb{Q}(0), \bigotimes_{i=1}^4 M_{\omega_i}(-1)).$$

2. As discussed in Section 1.10, there are two types of cyclic elements, see (44) and Fig 33, which should be related to  $L(S^2 M_{\omega}, 3)$ . Let us consider first the motivic correlator map assigned to the left one:

$$\mathrm{Cor}_{\mathcal{M}}^0 : \mathcal{C} \left( \widehat{\mathrm{Meas}(\mathrm{Cusps})}(-1)^{\otimes 2} \otimes S^2(M_{\omega}^{\vee} \otimes V_{\omega})(1) \right) \longrightarrow \mathcal{L}_{\mathcal{M}}.$$

By the Manin-Drinfeld theorem the components of the coproduct corresponding to the  $s$ -decorated vertices are zero. Furthermore, the only nontrivial component of the coproduct is the one corresponding to the cut shown on Fig 33 by a dotted line. It is given by

$$\sum_i \mathrm{Cor}_{\mathcal{M}}^0(s_1 \otimes \omega_1 \otimes \psi_i(1)) \otimes \mathrm{Cor}_{\mathcal{M}}^0(\psi_i^{\vee} \otimes \omega_2 \otimes s_2(1)) \quad (217)$$

To formulate a criteria for this element to be zero, we proceed as follows:

i) Since  $s_i$  are cuspidal divisors on a modular curve, each of the factors determines an element of  $\mathrm{Ext}_{\mathrm{Mot}}^1$ , e.g.  $\mathrm{Cor}_{\mathcal{M}}^0(s_1 \otimes \omega_1 \otimes \psi_i(1)) \in \mathrm{Ext}_{\mathrm{Mot}}^1(\mathbb{Q}(0), M_{\omega_1} \otimes M_{\psi_i})$ .

ii) We calculate the corresponding tensor product of the Hodge correlators, given explicitly by the Rankin-Selberg integrals:

$$\sum_i \mathrm{Cor}_{\mathcal{H}}^0(s_1 \otimes \omega_1 \otimes \psi_i(1)) \otimes \mathrm{Cor}_{\mathcal{H}}^0(\psi_i^{\vee} \otimes \omega_2 \otimes s_2(1)) \in \mathbb{C} \otimes \mathbb{C}. \quad (218)$$

The conjectural injectivity of the regulator map implies that (218) vanishes if and only if the coproduct (217) is zero.

The cyclic element provided by the right diagram on Fig 33 is treated similarly. This completes the formulation of our conjecture relating  $L(\mathrm{Sym}^2 f, 3)$  to the Hodge correlators.

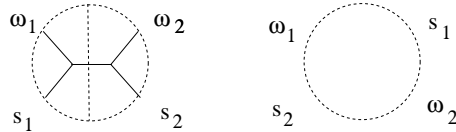


Figure 33: Correlators of type  $S^2 M_{\omega}(1)$ .

3. It would be interesting to generalize this picture, providing an automorphic description of a part of the motivic Galois group of the category of mixed motives over an arithmetic field  $F$ . Here is an example provided by the Drinfeld modular curves [Dr2] in the function field case.

**$l$ -adic correlators for the universal Drinfeld modular curve.** Let  $F$  be the function field of a curve  $X$  over a finite field. Choose a point  $x$  on  $X$  and take the Drinfeld modular curve parametrising the rank 2 elliptic modules corresponding to the affine curve  $X - x$  [Dr2]. Then there is the same kind of picture as above. Namely, one should have a conjectural abelian category of mixed motives over  $F$ . One can define its  $l$ -adic realization: a category of motivic

$l$ -adic Galois representations of the Galois group  $\text{Gal}(\overline{F}/F)$ , where  $l \neq \text{char}(F)$ . An example is provided by the  $l$ -adic fundamental Lie algebra of the Drinfeld modular curve. It is a free pronilpotent Lie algebra generated by  $H_1$  of the curve. There is a decomposition similar to (208). Thus, just as above, we arrive at the  $l$ -adic correlator map.

## 12 Feynman integral for the Hodge correlators

### 12.1 The Feynman integral

Let  $\varphi$  be a smooth function on  $X(\mathbb{C})$  with values in  $N \times N$  complex matrices  $\text{Mat}_N(\mathbb{C})$ . We say that  $\{\varphi\}$  is our space of fields. Let us choose a cyclic word

$$W = \mathcal{C}\left(\{a_0\} \otimes \omega_1^0 \otimes \dots \otimes \omega_{n_0}^0 \otimes \dots \otimes \{a_m\} \otimes \omega_1^m \otimes \dots \otimes \omega_{n_m}^m\right) \in \mathcal{CT}(V_{X,S}^\vee) \quad (219)$$

where  $a_i$  are points of  $X(\mathbb{C})$ , not necessarily different, and  $\omega_j^i \in \Omega_X^1 \oplus \overline{\Omega}_X^1$ . We are going to assign to  $W$  a functional  $\mathcal{F}_W(\varphi)$  on the space of fields.

There are the following elementary matrix-valued functionals on the space of fields:

(i) Each point  $a \in X(\mathbb{C})$  provides a functional

$$\mathcal{F}_a(\varphi) := \varphi(a) \in \text{Mat}_N(\mathbb{C}). \quad (220)$$

(ii) A 1-form  $\omega$  on  $X(\mathbb{C})$  provides a functional

$$\mathcal{F}_\omega(\varphi) := \int_{X(\mathbb{C})} [\varphi, d^c \varphi] \wedge \omega \in \text{Mat}_N(\mathbb{C}). \quad (221)$$

(iii) A pair of 1-forms  $(\omega_1, \omega_2)$  on  $X(\mathbb{C})$  provides a functional

$$\mathcal{F}_{(\omega_1, \omega_2)}(\varphi) := \int_{X(\mathbb{C})} \varphi(x) \omega_1 \wedge \omega_2 \in \text{Mat}_N(\mathbb{C}). \quad (222)$$

Let us choose a collection  $\mathcal{P}$ , possibly empty, of consecutive pairs  $\{\omega_j^i, \omega_{j+1}^i\}$  of 1-forms entering the cyclic word  $W$ . Take a field  $\varphi$ , and go along  $W$ , assigning to every element  $\{a_i\}$  a matrix  $\mathcal{F}_{a_i}(\varphi)$ , to every 1-form  $\omega$  which does not enter to  $\mathcal{P}$  a matrix  $\mathcal{F}_\omega(\varphi)$ , and to every pair of forms  $\{\omega_j^i, \omega_{j+1}^i\}$  from the collection  $\mathcal{P}$  a matrix  $\mathcal{F}_{(\omega_j^i, \omega_{j+1}^i)}(\varphi)$ . This way for a given field  $\varphi$  we obtain a cyclic word in  $\text{Mat}_N(\mathbb{C})$ , so its trace is well defined, providing a complex-valued functional, denoted by  $\mathcal{F}_{W, \mathcal{P}}(\varphi)$ . The functional  $\mathcal{F}_W(\varphi)$  is obtained by taking the sum over all possible collections  $\mathcal{P}$ :

$$\mathcal{F}_W(\varphi) = \sum_{\mathcal{P}} \mathcal{F}_{W, \mathcal{P}}(\varphi).$$

**Example.** Let  $W = \mathcal{C}\left(\{s_0\} \otimes \{s_1\} \otimes \dots \otimes \{s_m\}\right)$ . Then we recover formula (26) for  $\mathcal{F}_W(\varphi)$ .

Given an integer  $N$ , we would like to define the correlator corresponding to  $W$  via a Feynman integral

$$\text{Cor}_{X, N, h, \mu}(W) := \int \mathcal{F}_W(\varphi) e^{iS(\varphi)} \mathcal{D}\varphi \quad (223)$$

$$S(\varphi) := \frac{1}{2\pi i} \int_{X(\mathbb{C})} \text{Tr}\left(\frac{1}{2} \partial\varphi \wedge \bar{\partial}\varphi + \frac{1}{6} \hbar \cdot \varphi [\partial\varphi, \bar{\partial}\varphi]\right).$$

Observe that  $S(\varphi)$  is real:  $\overline{S(\varphi)} = S(\overline{\varphi})$ . Unfortunately formula (223) does not have a precise mathematical sense. We understand it by postulating the perturbation series expansion with respect to a small parameter  $\hbar$ , using the standard Feynman rules, and then taking the leading term in the asymptotic expansion as  $N \rightarrow \infty, \hbar = N^{-1/2}$ . Before the implementation of this plan, let us recall the finite dimensional situation.

## 12.2 Feynman rules

This subsection surveys well-known results, mostly due to Feynman and t'Hooft, and is included for the convinience of the reader only.

**Feynman rules for finite-dimensional integrals.** We follow [Wit], Section 1.3. Let  $V$  be a finite dimensional real vector space and  $B(v, v) \in S^2(V^*)$  a positive definite quadratic bilinear symmetric form on  $V$ . Denote by  $dv$  an invariant Lebesgue measure on  $V$  normalized by

$$\int_V e^{-B(v,v)/2} dv = 1.$$

Let us consider a polynomial function on  $V$  given by an expansion

$$Q(v) = \sum_{m \geq 1} g_m Q_m(v)/m!, \quad Q_m(v) \in S^m(V^*)$$

where  $g_m$  are formal variables. Let  $f_i(v) \in V^*$ . Consider the following integral, the *correlator* of  $f_1, \dots, f_N$ :

$$\langle f_1, \dots, f_N \rangle := \int_V f_1(v) \dots f_N(v) e^{-B(v,v)/2 + Q(v)} dv \in \mathbb{C}[[g_1, g_2, \dots]],$$

where we expand  $e^{Q(v)}$  into a series, and then calculate each term separately. This integral is computed using Feynman graphs as follows. Let  $\mathbf{n} := \{n_1, n_2, \dots\}$  be any sequence of nonnegative integers which is eventually zero. Let  $G(N, \mathbf{n})$  be the set of equivalence classes of graphs which have  $N$  1-valent vertices labeled by  $1, \dots, N$ , and  $n_i$  unlabeled  $i$ -valent vertices,  $i \geq 1$ . The labeled vertices are called *external*, and unlabeled ones *internal* vertices.

To each internal vertex  $v$  of  $\Gamma$  of valency  $|v|$  we assign a tensor  $Q_{|v|}$ . To each external vertex labeled by  $i$  we assign a vector  $f_i$ . Taking the product over all vertices of  $\Gamma$ , we get a vector

$$\prod_{i=1}^N f_i \prod_v Q_{|v|}$$

in the tensor product of  $V^*$ 's over the set of all flags of  $G$ . Further, let  $B^{-1} \in S^2 V$  be the inverse form to  $B$ . To each edge  $e$  of the graph  $\Gamma$  we assign a tensor  $B_e^{-1}$  called the *propogator*. Then there is a vector in the the tensor product of  $V$ 's over the set of all flags of  $G$ :

$$\prod_{\text{edges } e \text{ of } \Gamma} B_e^{-1}.$$

Contracting these two vectors we get a complex number

$$F_\Gamma := \left\langle \prod_{\text{vertices } v \text{ of } \Gamma} Q_{|v|}, \prod_{\text{edges } e \text{ of } \Gamma} B_e^{-1} \right\rangle \in \mathbb{C}$$



where for external vertices  $i$  we put  $Q_{|i|} := f_i$ .

**Theorem 12.1** a) *One has*

$$\langle f_1, \dots, f_N \rangle = \sum_n \prod_i g_i^{n_i} \sum_{\Gamma \in G(N, \mathbf{n})} |\text{Aut}(\Gamma)|^{-1} F_\Gamma(f_1, \dots, f_n) \quad (224)$$

where  $\text{Aut}(\Gamma)$  denotes the group of automorphisms of  $\Gamma$  which fix the external vertices.

b) *One has*

$$\log \langle f_1, \dots, f_N \rangle = \sum_n \prod_i g_i^{n_i} \sum_{\Gamma \in G_0(N, \mathbf{n})} |\text{Aut}(\Gamma)|^{-1} F_\Gamma(f_1, \dots, f_n) \quad (225)$$

where the sum now is over the subset  $G_0(N, \mathbf{n}) \subset G(N, \mathbf{n})$  of connected graphs only.

To deduce b) from a) observe that the sum in (224) is over all, possibly disconnected, diagrams. The automorphisms of a graph are given by the automorphisms of the connected components, and the permutation groups  $S_n$  of  $n$  identical copies of connected components. It remains to notice that

$$\prod_i \sum_j \frac{F_{\Gamma_i}^{n_j}}{n_j!} = \prod_i \exp(F_{\Gamma_i}) = \exp\left(\sum_i F_{\Gamma_i}\right).$$

**Feynman rules for the matrix model (223).** Let us explain how to write an asymptotic expansion in  $\hbar$  for the Feynman integral (223). The quadratic form in our case is

$$B(\varphi) = (2\pi i)^{-1} \sum_{i,j=1}^N \int_{X(\mathbb{C})} \partial \varphi_j^i \wedge \bar{\partial} \varphi_i^j = -(2\pi i)^{-1} \sum_{i,j=1}^N \int_{X(\mathbb{C})} \varphi_j^i \cdot \partial \bar{\partial} \varphi_i^j. \quad (226)$$

The vector space  $V$  is the infinite-dimensional space of  $\text{Mat}_N(\mathbb{C})$ -valued functions  $\varphi$  on  $X(\mathbb{C})$ .

Let us assume first that  $N = 1$ . Then the Laplacian  $\Delta = (2\pi i)^{-1} \bar{\partial} \partial$  has a one-dimensional kernel. So the inverse form  $B^{-1}$  can be defined only on a complement to the kernel. Such a complement is described by a choice of a 2-current  $\mu$  on  $X(\mathbb{C})$  with a non-zero integral, which we normalize to be 1. It consists of functions  $\varphi$  orthogonal to  $\mu$ . The Green function  $G_\mu(x, y)$  describes the bilinear form  $B^{-1}$  on the complement as follows. The dual  $V^*$  contains a dense subspace of smooth 2-forms  $\omega$  on  $X(\mathbb{C})$ . The value of the bilinear form on  $\omega_x \cdot \omega_y \in S^2 V^*$  is  $\int_{X(\mathbb{C})^2} G_\mu(x, y) \omega_x \otimes \omega_y$ . Its restriction to the subspace of  $\omega$ 's orthogonal to constants does not depend on the ambiguity in the definition of  $G_\mu(x, y)$ .

Now let  $N > 1$ . Let  $e_j^i$  be an elementary matrix with the only non-zero entry, 1, at the  $(i, j)$  place. Then it is clear from (226) that the propagator  $B^{-1}$  is given by

$$B^{-1} = -G_\mu(x, y) \sum_{i,j=1}^N e_j^i \otimes e_i^j.$$

Its Feynman diagram is on the left of Fig 34. Recall the cubical term of the action  $S(\varphi)$ :

$$\hbar \cdot \sum_{i,j,k=1}^N \int_{X(\mathbb{C})} \left( \varphi_j^i(x) \partial \varphi_k^j(x) \wedge \bar{\partial} \varphi_i^k(x) - \varphi_j^i(x) \bar{\partial} \varphi_k^j(x) \wedge \partial \varphi_i^k(x) \right).$$



Figure 34: Feynman diagrams for the propagator and the vertex contribution

Therefore the contribution of the three valent vertex shown on the right of Fig 34 is

$$\begin{aligned}
 & -\hbar \cdot \int_{X(\mathbb{C})} \left( G_\mu(a, x) \partial G_\mu(b, x) \wedge \bar{\partial} G_\mu(c, x) - G_\mu(a, x) \bar{\partial} G_\mu(b, x) \wedge \partial G_\mu(c, x) \right) \sum_{i,j,k=1}^N e_j^i \otimes e_k^j \otimes e_i^k = \\
 & \hbar \cdot \int_{X(\mathbb{C})} \omega_2^* \left( G_\mu(a, x) \wedge G_\mu(b, x) \wedge G_\mu(c, x) \right) \sum_{i,j,k=1}^N e_j^i \otimes e_k^j \otimes e_i^k. \quad (227)
 \end{aligned}$$

Integral (227) is convergent by Lemma 2.5. It does not change if we add a constant to  $G_\mu$ .

Recall that a ribbon graph is a graph with an additional structure: a cyclic order of the edges sharing each vertex. The contraction procedure for such propagator and vertex contributions implies that the non-zero contributions are given by ribbon graphs only (Fig 35). Each ribbon graph  $\Gamma$  contributes a Feynman integral entering with a certain weight  $w_\Gamma$ , calculated below.

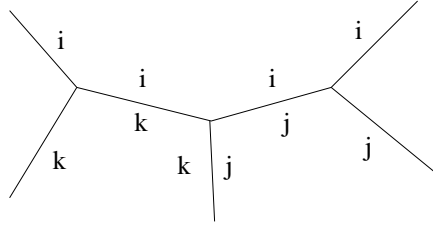


Figure 35: Non-zero contributions are given by ribbon graphs only.

A ribbon graph provides a surface  $S_\Gamma$ . It is obtained by taking a compact surface without boundary  $\bar{S}_\Gamma$  and removing from it several discs. The graph  $\Gamma$  is embedded into the surface  $S_\Gamma$ , so that the external vertices are on the boundary. It provides a decomposition of  $\bar{S}_\Gamma - \Gamma$  into polygons, called faces. Let  $f_\Gamma$ ,  $e_\Gamma$  and  $v_\Gamma$  be the number of polygons, edges and vertices of  $\Gamma$ . So each Feynman diagram  $\Gamma$  contributes an integral with the weight  $\omega_\Gamma = N^{f_\Gamma} \hbar^{v_\Gamma - (m+1)}$ .

The Euler characteristic of the surface  $\bar{S}_\Gamma$  is  $\chi_\Gamma = v_\Gamma + f_\Gamma - e_\Gamma$ . Let us shrink each boundary component of the surface  $S_\Gamma$  into a point. The resulting surface is homeomorphic to  $\bar{S}_\Gamma$ . It has inside a ribbon graph with all vertices of valency 3 except the shrunk points, whose total valency is  $m + 1$ . Thus  $(m + 1) + 3v_\Gamma = 2e_\Gamma$ . So  $\chi_\Gamma = f_\Gamma - v_\Gamma/2 - (m + 1)/2$ .

### 12.3 Hodge correlators and the matrix model

**Definition 12.2** *The correlator  $\text{Cor}_{X,\mu}(W)$  is the leading term of the asymptotics of  $\text{Cor}_{X,N,\mu}(W)$  as  $N \rightarrow \infty$  and  $\hbar = N^{-1/2}$ .*

The number of ribbon graphs decorated by a given cyclic word  $W$ , even if we restrict our attention to connected diagrams, is infinite: one could have a surface  $S_\Gamma$  of arbitrary genus, and even if the genus is zero, one could have as many loops in  $\Gamma$  as we want. However Lemma 12.3 tells that all but finitely many of them give zero contribution to  $\text{Cor}_X(W)$ . Precisely, only trees will contribute, and, of course, there are finitely many trees with given external vertices.

**Lemma 12.3** *The logarithm  $\log \text{Cor}_{X,\mu}(W)$  of the Feynman integral correlator is the sum of integrals assigned to connected Feynman diagrams corresponding to genus zero surfaces.*

**Proof.** Since  $\hbar = N^{-1/2}$ , the finite-dimensional integral corresponding to a Feynman diagram  $\Gamma$  enters with the weight  $\omega_\Gamma = N^{f_\Gamma - v_\Gamma/2 + (m+1)/2} = N^{(m+1) + \chi_\Gamma}$ . Thus the leading term of the asymptotics comes from the Feynman diagrams corresponding to the genus zero surfaces.

**Theorem 12.4** *The logarithm  $\log \text{Cor}_{X,\mu}(W)$  of the Feynman integral correlator equals, up to a sign computed below, to the value of the Hodge correlator  $\text{Cor}_\mathcal{H}^*(W)$  on  $W$ .*

**Proof.** We need to show that the contribution of a  $W$ -decorated plane trivalent tree  $T$  to the Hodge correlator is the same as the finite dimensional Feynman integral  $\varphi_T$  assigned by the Feynman rules to the Feynman diagram corresponding to  $T$ . The contribution to  $\varphi_T$  of a 3-valent vertex  $v$  of  $T$  is  $N^3$  times the integral

$$- \int_{X(\mathbb{C})} \left( G_\mu(a, x) \partial G_\mu(b, x) \wedge \bar{\partial} G_\mu(c, x) - G_\mu(a, x) \bar{\partial} G_\mu(b, x) \wedge \partial G_\mu(c, x) \right) = \quad (228)$$

$$\int_{X(\mathbb{C})} \omega_2^* \left( G_\mu(a, x) \wedge G_\mu(b, x) \wedge G_\mu(c, x) \right).$$

Integral (228) is symmetric under the permutations of  $(a, b, c)$ . Let us use this symmetry to find a preferred presentation for the integrand of the Feynman integral  $\varphi_T$  related to  $T$ .

Recall that a flag at a vertex  $v$  is a pair (the vertex  $v$ , an edge incident to  $v$ ). So each internal vertex  $v$  of  $T$  gives rise to three flags. We say that among the three flags  $(x, a), (x, b), (x, c)$  relevant to integral (228) the flag  $(x, a)$  is the *free flag at the vertex  $x$*  – it contributes the Green function  $G_\mu(a, x)$  rather than its derivative. Thanks to the symmetry, for any 3-valent vertex  $v$  of  $T$ , in the contribution to the Feynman integral  $\varphi_T$  we can make any flag incident to  $v$  to be the free flag. Choose an external edge of  $T$ , called the *root edge*. Then for each internal vertex there is a unique flag which is the closest to the root. We call it the *upper flag*. Using the symmetry of the integral (228) one easily proves the following:

**Lemma 12.5** *Given a root edge of  $T$ , there is a unique presentation of the integrand for the integral  $\varphi_T$  for which the set of free flags is the set of upper flags.*

On the other hand this contribution can be written as

$$2^{m-1} \int_{X(\mathbb{C})^{m-1}} G_0(\partial G_1 \wedge \bar{\partial} G_2) \wedge \dots \wedge (\partial G_{2m-3} \wedge \bar{\partial} G_{2m-2}), \quad (229)$$

where the integration is over the copies of  $X$  assigned to the internal vertices of  $T$ , and  $(G_{2k+1}, G_{2k+2})$  is the pair of Green functions assigned to the pair of non-upper edges of  $k$ -th vertex of  $T$ . We claim that

$$(229) = (-1)^{(m-1)(m-2)/2} \int_{X(\mathbb{C})^{m-1}} \omega_{2m-2}^*(G_0, \dots, G_{2m-2}).$$

Indeed, comparing (133) with the comment right after (135), we conclude that this is true up to a sign. Theorem 12.4 is proved.

**Remark.** The correlators corresponding to the Feynman diagrams with loops are divergent. A simplest example is shown on Fig 36. The corresponding correlator integral is

$$\int_{X(\mathbb{C}) \times X(\mathbb{C})} G(a, x) \wedge \partial_x \bar{\partial}_y G(x, y) \wedge \bar{\partial}_x G(x, y) \wedge \partial_y G(y, b).$$

I do not know whether the theory is renormalizable.

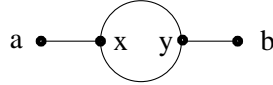


Figure 36: A Feynman diagram which gives rise to a divergent correlator.

On the other hand, the procedure which we use in Section 2 to define the Hodge correlators also does not make sense for diagrams with loops. Indeed, it produces a (non-closed) differential form whose degree is smaller than the real dimension of the integration cycle. For example for the diagram on Fig 36 we have 4 edges, and hence get a differential 3-form on  $X(\mathbb{C})^2$ .

**A generalization of the Feynman integral.** Let  $\mathcal{G}$  be a Lie algebra with an invariant symmetric bilinear form  $Q : \mathcal{G} \otimes \mathcal{G} \rightarrow \mathbb{C}$ . Consider  $\mathcal{G}$ -valued fields  $\{\varphi\}$  on  $X(\mathbb{C})$ . Then there is an action

$$S_{\mathcal{G}}(\varphi) := \frac{1}{2\pi i} \int_{X(\mathbb{C})} Q \left( \frac{1}{2} \partial \varphi \wedge \bar{\partial} \varphi + \frac{1}{6} \hbar \cdot \varphi \otimes [\partial \varphi, \bar{\partial} \varphi] \right).$$

Here the expression in parenthesis is a  $\mathcal{G} \otimes \mathcal{G}$ -valued 2-form on  $X(\mathbb{C})$ , so applying the bilinear form  $Q$  we get a 2-form and integrate it over  $X(\mathbb{C})$ . Formulas (220)-(221) provide  $\mathcal{G}$ -valued functionals on the space of fields corresponding to a choice of a point  $a$ , a 1-form, and a pair of 1-forms on  $X(\mathbb{C})$ . Repeating verbatim our construction we assign to a cyclic word  $W$  a correlator

$$\text{Cor}_{X, \mathcal{G}, \hbar}(W) := \int \mathcal{F}_W(\varphi) e^{i S_{\mathcal{G}}(\varphi)} \mathcal{D}\varphi. \quad (230)$$

The motion equation for the action is  $\delta S_{\mathcal{G}}(\varphi) / \delta \varphi = 0$ . Renormalising  $\varphi \mapsto \varphi / \hbar$  to remove dependence on  $\hbar$ , we get  $\bar{\partial} \partial \varphi = [\bar{\partial} \varphi, \partial \varphi]$ .

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